Bell Correlations from Scalar-Time Spectral Closure: Emergent SU(2) Measurement Geometry in Time–Scalar Field Theory

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May 11, 2026

Abstract

Bell correlations represent one of the deepest structural features of quantum mechanics. Experimental violations of Bell inequalities demonstrate that observed two-state correlations cannot be reproduced by classical local hidden-variable probability spaces satisfying Kolmogorov factorization constraints. Standard quantum mechanics accounts for these violations through Hilbert-space geometry, projector-valued measurements, and noncommuting observables. However, the geometric origin of this structure is normally postulated rather than derived.

In this paper we investigate whether the measurement geometry underlying Bell correlations can emerge naturally from the spectral closure structure of Time–Scalar Field Theory (TSFT). The goal is not to replace quantum mechanics, nor to construct a classical hidden-variable model reproducing Bell violations. Instead, the objective is narrower and mathematically sharper: to determine the conditions under which admissible scalar-time closure sectors become equivalent to quantum two-state measurement geometry.

Starting from the scalar-time field $\Theta(x^\mu)$ and its associated fluctuation operator, we construct a self-adjoint spectral problem governing admissible coherence modes. Under finite-energy, normalizability, and phase-return conditions, the admissible spectrum becomes discrete. We then examine degenerate two-dimensional closure eigenspaces and show that closure-preserving basis transformations naturally induce unitary rotational structure. After quotienting by physically irrelevant global phase, the admissible state geometry reduces to an emergent SU(2)-covariant two-state manifold.

Within this framework, binary measurements arise as closure-compatible projection operations on the degenerate sector. The associated correlation function acquires the form

$$E(a, b) = -a \cdot b,$$

matching the standard quantum singlet-state correlation law. Maximization of the corresponding Clauser–Horne–Shimony–Holt (CHSH) functional then yields the Tsirelson bound

$$S_{\max} = 2\sqrt{2},$$

as a geometric consequence of closure-preserving scalar-time measurement structure.

The derivation presented here does not assume Hilbert-space quantum mechanics as an initial axiom. Rather, the central claim is that under sufficiently constrained spectral closure conditions, the admissible geometry of scalar-time coherence sectors becomes mathematically equivalent to the two-state measurement geometry responsible for Bell correlations.

1 Introduction

Bell's theorem occupies a foundational position in modern physics because it sharply distinguishes classical probabilistic geometry from the experimentally observed structure of quantum correlations. Beginning from the Einstein–Podolsky–Rosen paradox, Bell demonstrated that no locally factorized hidden-variable theory satisfying classical probability constraints can reproduce the full correlation structure predicted by quantum mechanics and subsequently verified experimentally.

The significance of Bell's result is frequently overstated philosophically and understated geometrically. Bell's theorem does not prove that physical reality is intrinsically "mystical," nor does it directly identify a preferred ontology. Rather, it establishes that experimentally observed two-state correlations are incompatible with a classical Kolmogorov probability space built from simultaneously well-defined local hidden variables obeying factorized measurement independence.

Standard quantum mechanics resolves this conflict through Hilbert-space geometry. Physical states are represented as rays in a complex vector space, observables correspond to self-adjoint operators, and binary measurements arise through projector-valued decompositions. For maximally entangled two-state systems, this structure yields the correlation law

$$E(a,b) = -\cos\theta,$$

and consequently the Tsirelson bound

$$S_{\max} = 2\sqrt{2}.$$

However, in conventional quantum mechanics the underlying measurement geometry is postulated rather than derived. Hilbert-space structure, unitary symmetry, and projector rules are introduced axiomatically. The deeper origin of the corresponding state geometry therefore remains open.

Time–Scalar Field Theory (TSFT) approaches this question from a different direction. Rather than assuming quantum structure as fundamental, TSFT proposes that admissible physical states arise as stable coherence structures within an underlying scalar-time field $\Theta(x^\mu)$. Prior work within the TSFT program has developed:

(1) scalar-time dynamical actions,

(2) self-adjoint fluctuation operators,

(3) spectral closure conditions,

(4) SU(N)-covariant sectorization,

(5) first-order spinor factorization,

(6) uncertainty structure from Weyl-pair geometry,

(7) and Born-type projector measures associated with closure amplitudes.

The present paper investigates whether these ingredients are sufficient to recover the specific two-state geometry responsible for Bell correlations.

Importantly, the goal is not to construct a local hidden-variable alternative to quantum mechanics. Bell-type local realism models have repeatedly failed because classical probability geometry is too restrictive to reproduce experimentally observed correlation structure. The objective here is fundamentally different. We instead ask whether nonclassical measurement geometry itself may emerge naturally from closure-preserving scalar-time spectral structure.

More precisely, the central question becomes:

> Under what conditions does an admissible degenerate scalar-time closure sector become mathematically equivalent to a quantum two-state measurement manifold?

The strategy pursued in this paper is therefore structural rather than interpretive. Beginning from the scalar-time fluctuation operator, we derive a discrete admissible spectrum under finite-energy and phase-return conditions. We then analyze degenerate two-dimensional eigenspaces and examine the transformations preserving closure structure within those sectors. We show that closure-preserving indistinguishability naturally induces unitary rotational freedom. Modulo global phase redundancy, the resulting geometry becomes SU(2)-covariant.

Once this geometry is established, projector-valued binary measurements arise naturally as admissible closure decompositions. The Bell correlation structure then follows as a direct geometric consequence rather than an independent axiom.

The structure of the paper is as follows. Section II introduces the scalar-time fluctuation framework and derives the closure operator. Section III establishes the spectral closure conditions leading to discrete admissible eigenspaces. Section IV analyzes degenerate closure sectors and derives the emergence of SU(2)-covariant rotational structure. Section V constructs closure-compatible binary measurement projectors. Section VI derives the Bell correlation law and the CHSH Tsirelson bound. Section VII discusses implications, limitations, and directions for further development.

Recent decades have also seen substantial work in quantum reconstruction programs deriving aspects of quantum theory from operational, informational, or geometric axioms, including the approaches of Hardy, Masanes and Müller, and Chiribella, D'Ariano, and Perinotti [4, 5, 6]. The present approach differs structurally from these programs by beginning not from operational axioms but from scalar-time spectral closure dynamics and recurrence-compatible admissibility conditions.

The central claim advanced here is therefore narrow but precise. We do not claim to derive all of quantum mechanics from scalar-time structure. Rather, we show that under sufficiently constrained spectral closure conditions, the admissible geometry of scalar-time coherence sectors becomes mathematically equivalent to the two-state measurement geometry responsible for Bell correlations.

2 Linearized Scalar-Time Closure Dynamics

We now examine small fluctuations about a stable scalar-time background configuration and derive the spectral closure structure underlying admissible recurrent modes.

The goal of this section is to establish the operator framework from which closure-preserving degenerate sectors, projective geometry, and Bell-type correlations later emerge.

2.1 Background Scalar-Time Configuration

Let

$$\Theta(x^\mu)$$

be the scalar-time field governed by the action

$$S[\Theta] = \int d^4x \left[\frac{1}{2}\partial_\mu \Theta \, \partial^\mu \Theta - V(\Theta) \right].$$

Variation of the action gives the scalar-time field equation

$$\Box \Theta = V'(\Theta).$$

Suppose

$$\Theta_0(x)$$

is a stable background solution satisfying

$$\Box \Theta_0 = V'(\Theta_0).$$

We now study small perturbations about this background.

2.2 Linearized Fluctuation Expansion

Write

$$\Theta(x,t) = \Theta_0(x) + \psi(x,t),$$

where

$$|\psi| \ll 1.$$

Expanding the potential about the background gives

$$V(\Theta_0 + \psi) = V(\Theta_0) + V'(\Theta_0)\psi + \frac{1}{2}V''(\Theta_0)\psi^2 + O(\psi^3).$$

Since the background satisfies the scalar-time field equation, the linear term cancels. Retaining terms through quadratic order yields the fluctuation action

$$S^{(2)}[\psi] = \frac{1}{2}\int d^4x \left[\partial_\mu \psi \, \partial^\mu \psi - V''(\Theta_0)\psi^2 \right].$$

Variation gives the linearized fluctuation equation

$$\Box\psi - V''(\Theta_0)\psi = 0.$$

The covariant fluctuation dynamics therefore take the form

$$\left(-\Box + V''(\Theta_0)\right)\psi = 0.$$

2.3 Spatial-Slice Spectral Reduction

The preceding equation identifies the Lorentz-compatible fluctuation dynamics of the scalar-time field. However, the spectral closure analysis developed in the remainder of this paper is not performed on the full Lorentzian spacetime operator

$$-\Box + V''(\Theta_0).$$

That operator is hyperbolic in Lorentzian signature and is therefore not the Schrödinger-type elliptic operator to which the standard self-adjoint spectral theory applies.

Instead, the closure spectrum is defined on a spatial Cauchy slice

$$\Sigma_t,$$

after separating scalar-time evolution from spatial mode structure.

For stationary or quasistationary backgrounds, write the fluctuation modes in separated form

$$\psi(x,\tau) = u(x)e^{-i\omega\tau},$$

where

$$x \in \Sigma_t$$

denotes spatial position and

$$\tau$$

is the scalar-time evolution parameter.

The spatial closure modes are then governed by the elliptic operator

$$L_\Theta u = \left[-\nabla^2 + V''(\Theta_0)\right]u = \lambda u,$$

acting on

$$L^2(\Sigma_t, d^3x).$$

The inner product used throughout the closure spectral construction is therefore

$$\langle u, v\rangle = \int_{\Sigma_t} d^3x\, u^*(x)v(x).$$

Finite norm means

$$\int_{\Sigma_t} d^3x\, |u(x)|^2 < \infty,$$

and finite closure energy means

$$E[u] = \int_{\Sigma_t} d^3x \left(|\nabla u|^2 + V''(\Theta_0)|u|^2 \right) < \infty.$$

Under the usual regularity, boundary, and lower-boundedness assumptions on $V''(\Theta_0)$, the spatial operator

$$L_\Theta = -\nabla^2 + V''(\Theta_0)$$

is a Schrödinger-type operator admitting a self-adjoint realization on

$$L^2(\Sigma_t).$$

Accordingly, the self-adjoint spectral theory used below applies to the spatial closure operator on Σ_t, while Lorentz compatibility remains encoded in the underlying scalar-time field equation.

2.4 Self-Adjointness and Closure Evolution

Let

$$L_\Theta = -\nabla^2 + V''(\Theta_0)$$

act on a dense domain

$$D(L_\Theta) \subset L^2(\Sigma_t).$$

Assuming the potential

$$V''(\Theta_0)$$

is real-valued and sufficiently regular, integration by parts gives

$$\langle u, L_\Theta v \rangle = \langle L_\Theta u, v \rangle$$

for admissible functions satisfying the appropriate boundary conditions.

Thus

$$L_\Theta$$

is symmetric. Under the standard lower-boundedness and regularity assumptions on the potential, the operator admits a self-adjoint realization.

Consequently, Stone's theorem implies that the scalar-time evolution operator

$$U(\tau) = e^{-iL_\Theta \tau}$$

forms a one-parameter unitary group on

$$L^2(\Sigma_t).$$

Therefore closure evolution preserves:

1. normalization,
2. orthogonality,
3. spectral decomposition,
4. and recurrence-compatible phase structure.

2.5 Closure Eigenmodes

The admissible closure modes satisfy the spectral equation

$$L_\Theta u_n = \lambda_n u_n.$$

The corresponding scalar-time evolution is

$$u_n(x, \tau) = u_n(x) e^{-i\lambda_n \tau}.$$

Thus each admissible closure mode possesses:

1. a spatial coherence structure,
2. a recurrent scalar-time phase,
3. and a conserved closure norm under unitary evolution.

The physically admissible closure sectors developed below arise from the recurrence and degeneracy structure of these scalar-time spectral modes.

3 Spectral Closure and Discrete Admissible Coherence Modes

The existence of a self-adjoint fluctuation operator alone does not yet produce physically admissible states. A generic self-adjoint operator may possess continuous spectra, unstable modes, non-returning phase evolution, or non-normalizable excitations. The central question is therefore not merely spectral existence, but spectral closure.

Within the Time–Scalar Field Theory framework, admissible physical structures are interpreted as coherence modes capable of maintaining dynamically stable propagation under scalar-time evolution. Modes that disperse indefinitely, fail normalization, or do not maintain coherent phase return are excluded from the physically admissible sector.

The purpose of the present section is to formalize these closure conditions and derive the resulting discrete admissible spectrum.

3.1 Closure Admissibility Conditions

Not every formal solution of the scalar-time fluctuation equation corresponds to a physically admissible closure mode. The scalar-time closure framework instead selects recurrent, finite-energy, coherence-stable spectral sectors.

We therefore impose the following admissibility conditions on closure states.

3.1.1 Finite Closure Energy

Let

$$u(x) \in L^2(\Sigma_t)$$

be a spatial closure mode satisfying

$$L_\Theta u = \lambda u,$$

where

$$L_\Theta = -\nabla^2 + V''(\Theta_0).$$

The associated closure energy functional is

$$E[u] = \int_{\Sigma_t} d^3x \left(|\nabla u|^2 + V''(\Theta_0)|u|^2 \right).$$

Physically admissible closure modes require

$$E[u] < \infty.$$

This excludes spatially divergent or nonlocalized configurations incapable of maintaining stable scalar-time coherence.

3.1.2 Normalizability

The closure norm is defined by

$$\|u\|^2 = \langle u, u \rangle = \int_{\Sigma_t} d^3x \, |u(x)|^2.$$

Admissible closure modes satisfy

$$\|u\|^2 < \infty.$$

Normalizability ensures that closure amplitudes remain finite and that physically meaningful transition probabilities may be consistently defined within the closure sector.

3.1.3 Recurrence-Compatible Phase Evolution

Let

$$U(\tau) = e^{-iL_\Theta \tau}$$

be the scalar-time evolution operator generated by the self-adjoint closure operator

$$L_\Theta.$$

A closure state

$$u \in L^2(\Sigma_t)$$

is recurrence-compatible if there exists a sequence

$$\tau_n \to \infty$$

and phases

$$\phi_n \in \mathbb{R}$$

such that

$$\left\| U(\tau_n)u - e^{i\phi_n}u \right\| \to 0.$$

Thus admissible closure modes are not required to exhibit exact finite-period recurrence. Instead, they must possess asymptotic phase-return structure compatible with recurrent scalar-time coherence.

This condition corresponds to almost-periodic recurrence in the sense of the spectral closure framework developed in Appendix G.

3.1.4 Closure Stability

Closure-admissible sectors must remain stable under scalar-time evolution.

If

$$u(x,\tau) = U(\tau)u(x,0),$$

then admissibility requires preservation of:

1. finite norm,
2. finite closure energy,
3. recurrence-compatible phase structure,
4. and bounded spectral evolution.

States exhibiting unbounded dispersive growth or irreversible dephasing are therefore excluded from the admissible closure sector.

3.1.5 Physical Interpretation

The closure admissibility conditions select physically sustainable scalar-time coherence structures.

Specifically:

1. finite energy excludes divergent closure configurations,

2. normalizability ensures finite closure amplitude,
3. recurrence-compatible evolution preserves scalar-time phase structure,
4. and closure stability excludes incoherent spectral sectors.

Consequently, admissible closure modes correspond to recurrent coherence-stable scalar-time spectral structures capable of supporting persistent physical observables.

3.2 Discrete Closure Sectors

The admissibility conditions developed above strongly constrain the spectral structure of physically sustainable scalar-time closure modes.

Let

$$u(\tau) = U(\tau)u(0) = e^{-iL_\Theta \tau}u(0)$$

be scalar-time evolution generated by the self-adjoint closure operator

$$L_\Theta.$$

If

$$u = \sum_n c_n u_n$$

belongs to a discrete spectral sector, then

$$u(\tau) = \sum_n c_n e^{-i\lambda_n \tau} u_n.$$

The corresponding overlap amplitude is

$$A(\tau) = \langle u, u(\tau)\rangle = \sum_n |c_n|^2 e^{-i\lambda_n \tau}.$$

Because the evolution is a superposition of recurrent phase rotations, discrete spectral sectors naturally support almost-periodic scalar-time recurrence and therefore satisfy the closure admissibility conditions developed in Appendix G.

By contrast, absolutely continuous spectral sectors generically fail recurrence-compatible phase return because nearby spectral components dephase under long scalar-time evolution.

Consequently, the physically admissible recurrent sector is discrete or effectively discrete under the closure admissibility conditions.

The admissible closure framework therefore selects recurrence-compatible coherence sectors capable of supporting:

1. persistent scalar-time phase structure,
2. stable closure amplitudes,
3. and long-lived coherence-preserving observables.

These recurrent sectors form the spectral foundation for the degenerate closure geometry developed below.

3.3 Closure Orthogonality

Because L_Θ is self-adjoint, distinct admissible eigenmodes satisfy

$$\langle \psi_m, \psi_n \rangle = 0, \qquad m \neq n.$$

After normalization,

$$\langle \psi_n, \psi_n \rangle = 1.$$

Thus the admissible closure sector possesses an orthonormal spectral structure.

Importantly, however, this orthonormality has not been postulated as a Hilbert-space axiom. It arises dynamically from:

1. self-adjoint fluctuation structure,
2. finite-energy closure,
3. normalizability,
4. and phase-return admissibility.

3.4 Degenerate Closure Sectors

The emergence of Bell-type geometry requires more than isolated discrete modes. It requires internal rotational freedom between admissible states.

We therefore focus on degenerate closure eigenspaces.

Suppose

$$L_\Theta \psi_1 = \lambda \psi_1, \qquad L_\Theta \psi_2 = \lambda \psi_2,$$

with

$$\langle \psi_1, \psi_2 \rangle = 0.$$

Then any linear combination

$$\Psi = c_1 \psi_1 + c_2 \psi_2$$

also satisfies

$$L_\Theta \Psi = \lambda \Psi.$$

Thus degeneracy induces an internal closure freedom:

$$(\psi_1, \psi_2) \longrightarrow (c_1, c_2).$$

This observation is the first indication that admissible closure sectors possess intrinsic rotational structure.

However, rotational freedom alone does not yet imply quantum geometry. The crucial remaining question is:

> What transformations preserve physically admissible closure structure inside a degenerate coherence sector?

The next section derives the answer directly from closure-preserving indistinguishability and shows that the admissible transformation group becomes unitary, yielding emergent SU(2)-covariant state geometry.

4 Closure-Preserving Transformations and Emergent SU(2) Geometry

The previous section established that degenerate scalar-time closure eigenspaces possess internal linear freedom. The present section determines the admissible transformations acting within such sectors.

The central result is that closure-preserving indistinguishability forces the admissible transformation group to be unitary. After quotienting by physically irrelevant global phase, the resulting state geometry becomes SU(2)-covariant.

This structure is not postulated. It emerges from the closure conditions already established.

4.1 Degenerate Closure Sector

Let

$$L_\Theta \psi_1 = \lambda \psi_1, \qquad L_\Theta \psi_2 = \lambda \psi_2,$$

with

$$\langle \psi_i, \psi_j \rangle = \delta_{ij}.$$

The degenerate closure sector is therefore

$$\mathcal{H}_\lambda = \mathrm{span}\{\psi_1, \psi_2\}.$$

Any admissible state in this sector may be written

$$\Psi = c_1 \psi_1 + c_2 \psi_2,$$

with complex coefficients c_1, c_2.

Because both basis states share the same closure eigenvalue, scalar-time evolution acts identically on the entire sector:

$$U(\tau)\Psi = e^{-i\lambda\tau}\Psi.$$

Thus relative structure inside the degenerate sector is preserved under scalar-time propagation.

4.2 Closure-Indistinguishable Bases

Suppose we replace the basis

$$(\psi_1, \psi_2)$$

with another basis

$$(\psi_1', \psi_2')$$

defined by

$$\psi_i' = \sum_{j=1}^{2} U_{ij}\psi_j.$$

We now ask:

Which transformations preserve physically admissible closure structure?

The admissibility conditions derived previously require preservation of:

1. finite norm,
2. orthogonality,
3. closure amplitude,
4. and phase-return consistency.

In particular, physically observable closure probabilities must remain invariant under basis relabeling inside the degenerate sector.

Therefore admissible transformations must preserve the inner product

$$\langle \Psi, \Phi \rangle.$$

For arbitrary states

$$\Psi = \sum_i c_i\psi_i, \qquad \Phi = \sum_i d_i\psi_i,$$

the transformed coefficient vectors satisfy

$$c_i' = \sum_j U_{ij}c_j, \qquad d_i' = \sum_j U_{ij}d_j.$$

Equivalently,

$$c' = Uc, \qquad d' = Ud.$$

The transformed inner product becomes

$$\langle \Psi', \Phi' \rangle = (c')^\dagger d' = (Uc)^\dagger (Ud) = c^\dagger U^\dagger U d.$$

Closure-preserving indistinguishability therefore requires

$$\langle \Psi', \Phi' \rangle = \langle \Psi, \Phi \rangle$$

for arbitrary admissible states.

Hence

$$U^{\dagger}U = I.$$

Therefore the admissible transformation group is unitary.

Proposition 1. *Closure-preserving transformations acting within a degenerate scalar-time coherence sector form a unitary group.*

This conclusion follows solely from:

1. spectral degeneracy,
2. closure preservation,
3. and basis indistinguishability.

No Hilbert-space axiom has been independently assumed.

4.3 Global Phase Redundancy

Physical closure observables are insensitive to overall global phase.

Indeed,

$$\Psi \longrightarrow e^{i\alpha}\Psi$$

leaves:

$$|\Psi|^2, \qquad \langle\Psi, \Psi\rangle, \qquad \text{and all relative closure amplitudes}$$

unchanged.

Therefore physically distinguishable states correspond not to vectors themselves, but to equivalence classes modulo global phase.

The physical state manifold is therefore the projective quotient

$$\mathbb{CP}^1 \cong S^2.$$

The admissible closure symmetry group acting on this manifold is

$$SU(2),$$

which acts transitively on the projective closure sphere through closure-preserving unitary transformations.

Thus the physically relevant geometry is not the full vector space itself, but the projective two-state closure manifold generated by degenerate scalar-time coherence sectors.

4.4 Bloch-Type State Geometry

Normalized states satisfy

$$|c_1|^2 + |c_2|^2 = 1.$$

Modulo global phase, any state may therefore be written

$$\Psi(\theta, \phi) = \cos\frac{\theta}{2}\,\psi_1 + e^{i\phi}\sin\frac{\theta}{2}\,\psi_2.$$

Thus physically admissible closure states are parameterized by:

$$(\theta, \phi) \in S^2.$$

The resulting state manifold is geometrically identical to the Bloch sphere of ordinary two-state quantum mechanics.

Importantly, this structure has emerged from:

1. self-adjoint spectral closure,
2. degenerate admissible eigenspaces,
3. norm-preserving indistinguishability,
4. and global-phase redundancy.

At no point has quantum two-state geometry been independently postulated.

4.5 Physical Interpretation

The emergence of SU(2)-covariant geometry has a direct closure interpretation.

Within a degenerate scalar-time sector, admissible coherence modes cannot be physically distinguished by arbitrary internal basis rotations preserving closure norm and phase consistency.

Consequently:

1. admissible closure states form a rotational manifold,
2. physical observables depend only on relative closure orientation,
3. and measurement structure becomes intrinsically geometric rather than classically set-theoretic.

This is precisely the structural transition required to escape classical Bell-type probability constraints.

Classical hidden-variable models assume that observables correspond to simultaneously well-defined elements of an underlying Kolmogorov probability space.

By contrast, the present framework yields observables as projections on a constrained closure manifold whose admissible decompositions depend on measurement orientation within the SU(2)-covariant state geometry.

The next section constructs these closure-compatible measurement projectors explicitly and derives the corresponding probability structure.

5 Closure-Compatible Measurement Projectors and Probability Structure

The previous section established that admissible degenerate scalar-time closure sectors possess SU(2)-covariant state geometry. The present section derives the corresponding measurement structure.

The central objective is to determine how binary observables arise within the closure framework and to derive the associated probability law directly from closure amplitudes rather than postulating the Born rule independently.

5.1 Binary Closure Measurements

Let

$$\mathcal{H}_\lambda = \mathrm{span}\{\psi_1, \psi_2\}$$

be a normalized degenerate closure sector.

A binary measurement corresponds to resolving the state relative to a chosen closure orientation inside the admissible SU(2) manifold.

For a measurement direction a, define the associated normalized closure basis

$$\psi_a^{(+)}, \qquad \psi_a^{(-)},$$

satisfying

$$\langle \psi_a^{(+)}, \psi_a^{(+)} \rangle = 1,$$

$$\langle \psi_a^{(-)}, \psi_a^{(-)} \rangle = 1,$$

and

$$\langle \psi_a^{(+)}, \psi_a^{(-)} \rangle = 0.$$

The associated closure projectors are

$$P_a^{(+)} = \left| \psi_a^{(+)} \right\rangle \left\langle \psi_a^{(+)} \right|,$$

$$P_a^{(-)} = \left| \psi_a^{(-)} \right\rangle \left\langle \psi_a^{(-)} \right|.$$

These satisfy:

$$P_a^{(+)} + P_a^{(-)} = I,$$

$$(P_a^{(\pm)})^2 = P_a^{(\pm)},$$

and

$$P_a^{(+)} P_a^{(-)} = 0.$$

Thus binary measurements arise as orthogonal closure decompositions of the admissible scalar-time sector.

5.2 Closure Amplitudes

Let the system occupy normalized closure state

$$\Psi.$$

The closure amplitude associated with outcome + along direction a is

$$A_a^{(+)} = \langle \psi_a^{(+)}, \Psi \rangle.$$

Similarly,

$$A_a^{(-)} = \langle \psi_a^{(-)}, \Psi \rangle.$$

Because the closure basis is orthonormal,

$$\Psi = A_a^{(+)} \psi_a^{(+)} + A_a^{(-)} \psi_a^{(-)}.$$

Normalization gives

$$|A_a^{(+)}|^2 + |A_a^{(-)}|^2 = 1.$$

5.3 Compatibility of Probability Weights with Closure Geometry

We now determine the form of probability assignments compatible with the admissible closure structure.

The closure framework requires:

1. positivity,
2. normalization,
3. basis covariance,
4. and invariance under global phase.

Global-phase invariance implies that physically observable outcome weights cannot depend on

$$A \quad \text{versus} \quad e^{i\alpha} A.$$

Therefore admissible probabilities may depend only on phase-invariant quantities constructed from closure amplitudes.

Moreover, closure-preserving unitary transformations leave overlap amplitudes invariant:

$$\langle U\Phi, U\Psi \rangle = \langle \Phi, \Psi \rangle.$$

Accordingly, quadratic overlap weights of the form

$$|\langle \Phi, \Psi \rangle|^2$$

are naturally compatible with:

1. closure covariance,
2. normalization,
3. and projective phase redundancy.

The detailed geometric derivation showing that the unique continuous SU(2)-covariant transition probability on the admissible two-state closure manifold takes the quadratic form

$$P(\Phi|\Psi) = |\langle \Phi, \Psi \rangle|^2$$

is developed in Appendix I using the Fubini–Study geometry of the projective closure manifold.

Accordingly, the closure-compatible binary probabilities are

$$P_a^{(+)} = |A_a^{(+)}|^2, \qquad P_a^{(-)} = |A_a^{(-)}|^2.$$

Equivalently,

$$P_a^{(+)} = |\langle \psi_a^{(+)}, \Psi \rangle|^2, \qquad P_a^{(-)} = |\langle \psi_a^{(-)}, \Psi \rangle|^2.$$

Thus the admissible scalar-time closure sector acquires the standard Born-type projection structure.

Proposition 2. *Within an admissible degenerate scalar-time closure sector, closure-compatible binary probabilities are given by squared projection amplitudes consistent with the projective geometry of the closure manifold.*

Importantly, the probability structure is not independently postulated. Rather, it emerges from:

1. closure-preserving unitary geometry,
2. projective phase reduction,
3. and the Fubini–Study structure of the admissible two-state closure manifold.

5.4 Spin-Type Measurement Operators

The SU(2)-covariant closure geometry permits the introduction of directional binary observables analogous to spin measurements.

Let

$$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

denote the Pauli generators acting on the degenerate closure sector.

For any unit vector

$$a \in \mathbb{R}^3, \qquad |a| = 1,$$

define the closure observable

$$\Sigma(a) = a \cdot \boldsymbol{\sigma}.$$

The eigenvalues of $\Sigma(a)$ are

$$\pm 1.$$

Thus binary closure measurements correspond to projections along directions on the emergent closure sphere.

The expectation value in state Ψ becomes

$$\langle \Sigma(a) \rangle_\Psi = \langle \Psi, \Sigma(a)\Psi \rangle.$$

At this stage the full geometric machinery required for Bell correlations is now present:

1. SU(2)-covariant state geometry,
2. closure-compatible projectors,
3. Born-type probabilities,
4. and directional binary observables.

The next section applies this structure to correlated two-sector closure states and derives the Bell correlation law together with the Tsirelson bound.

6 Bell Correlations and the Tsirelson Bound from Scalar-Time Closure Geometry

The previous sections established the emergence of:

1. discrete admissible closure sectors,
2. SU(2)-covariant two-state geometry,
3. closure-compatible projectors,
4. and Born-type probability structure.

We now examine correlated two-sector closure states and derive the Bell correlation law together with the Clauser–Horne–Shimony–Holt (CHSH) Tsirelson bound.

The key point is that the resulting nonclassical correlations arise from the geometry of admissible closure projections rather than from local hidden-variable factorization.

6.1 Two-Sector Closure State

Consider two identical degenerate scalar-time closure sectors:

$$\mathcal{H}_A, \qquad \mathcal{H}_B.$$

The combined admissible sector is

$$\mathcal{H}_A \otimes \mathcal{H}_B.$$

Let

$$\{\psi_+, \psi_-\}$$

denote an orthonormal basis for each sector.

We define the antisymmetric correlated closure state

$$\Psi_S = \frac{1}{\sqrt{2}} \left(\psi_+ \otimes \psi_- - \psi_- \otimes \psi_+ \right).$$

This state possesses rotational invariance under simultaneous SU(2) transformations acting on both sectors.

6.2 Directional Closure Observables

For measurement directions

$$a, b \in \mathbb{R}^3, \qquad |a| = |b| = 1,$$

define the binary closure observables

$$\Sigma_A(a) = a \cdot \boldsymbol{\sigma},$$

$$\Sigma_B(b) = b \cdot \boldsymbol{\sigma}.$$

The corresponding correlation function is

$$E(a, b) = \langle \Psi_S, \Sigma_A(a) \otimes \Sigma_B(b) \Psi_S \rangle.$$

6.3 Evaluation of the Correlation Function

Using the Pauli algebra

$$\sigma_i \sigma_j = \delta_{ij} I + i\epsilon_{ijk}\sigma_k,$$

together with antisymmetry of the correlated closure state, one obtains

$$\langle \Psi_S, \sigma_i \otimes \sigma_j \Psi_S \rangle = -\delta_{ij}.$$

Therefore

$$E(a, b) = -\sum_{i,j} a_i b_j \delta_{ij}.$$

Hence

$$E(a, b) = -a \cdot b.$$

Equivalently,

$$E(a, b) = -\cos\theta,$$

where θ is the angle between measurement directions.

Proposition 3. *Correlated degenerate scalar-time closure sectors possess the Bell correlation law*

$$E(a, b) = -a \cdot b.$$

Importantly, this correlation structure arises from:

1. closure-preserving SU(2) geometry,
2. projector-valued binary measurements,
3. and rotational covariance.

No local hidden-variable probability space has been introduced.

6.4 CHSH Functional

Define four measurement directions

$$a, a', b, b'.$$

The CHSH functional is

$$S = E(a, b) + E(a, b') + E(a', b) - E(a', b').$$

Classical local hidden-variable theories satisfy the Bell inequality

$$|S| \leq 2.$$

We now determine the maximal value permitted by scalar-time closure geometry.

6.5 Operator Form

Define the CHSH operator

$$\mathcal{B} = \Sigma_A(a) \otimes \Sigma_B(b) + \Sigma_A(a) \otimes \Sigma_B(b') + \Sigma_A(a') \otimes \Sigma_B(b) - \Sigma_A(a') \otimes \Sigma_B(b').$$

Equivalently, writing

$$A = a \cdot \sigma, \qquad A' = a' \cdot \sigma, \qquad B = b \cdot \sigma, \qquad B' = b' \cdot \sigma,$$

we obtain

$$\mathcal{B} = A \otimes (B + B') + A' \otimes (B - B').$$

Since

$$A^2 = A'^2 = B^2 = (B')^2 = I,$$

a direct expansion gives

$$\mathcal{B}^2 = 4I - [A, A'] \otimes [B, B'].$$

Using the Pauli commutator identities

$$[a \cdot \sigma, a' \cdot \sigma] = 2i(a \times a') \cdot \sigma,$$

and

$$[b\cdot\sigma, b'\cdot\sigma] = 2i(b\times b')\cdot\sigma,$$

we obtain

$$[A,A']\otimes[B,B'] = (2i)^2\big((a\times a')\cdot\sigma\big)\otimes\big((b\times b')\cdot\sigma\big).$$

Since

$$(2i)^2 = -4,$$

it follows that

$$\mathcal{B}^2 = 4I + 4\big((a\times a')\cdot\sigma\big)\otimes\big((b\times b')\cdot\sigma\big).$$

Taking operator norms,

$$\|\mathcal{B}^2\| \le 4 + 4\left\|(a\times a')\cdot\sigma\right\|\left\|(b\times b')\cdot\sigma\right\|.$$

For any vector

$$v\in\mathbb{R}^3,$$

the Pauli operator satisfies

$$\|v\cdot\sigma\| = |v|.$$

Therefore

$$\left\|(a\times a')\cdot\sigma\right\| = |a\times a'| \le 1,$$

and similarly

$$\left\|(b\times b')\cdot\sigma\right\| = |b\times b'| \le 1.$$

Hence

$$\|\mathcal{B}^2\| \le 4+4 = 8.$$

Therefore

$$\|\mathcal{B}\| \le 2\sqrt{2}.$$

For any normalized admissible closure state

$$\Psi,$$

we obtain

$$|\langle\Psi,\mathcal{B}\Psi\rangle| \le \|\mathcal{B}\| \le 2\sqrt{2}.$$

Thus the CHSH functional satisfies

$$|S| \le 2\sqrt{2}.$$

Therefore the maximal Bell correlation compatible with the admissible scalar-time closure geometry is

$$S_{\max} = 2\sqrt{2}.$$

6.6 Optimal Measurement Geometry

Equality is achieved for coplanar measurement directions separated by relative angles

$$45^\circ, \qquad 135^\circ.$$

Thus the Tsirelson bound is not arbitrary. It is the maximal correlation compatible with the rotational geometry of the closure manifold.

6.7 Interpretation

The derivation above does not reproduce Bell violations through classical hidden variables.

Instead, the nonclassical structure emerges because admissible observables correspond to projections on a constrained closure geometry rather than simultaneously well-defined elements of a classical probability space.

The scalar-time framework therefore recovers Bell correlations through:

1. self-adjoint spectral closure,
2. degenerate admissible eigenspaces,
3. closure-preserving unitary geometry,
4. projector-valued binary observables,
5. and rotational covariance.

The experimentally observed Bell structure appears not as an independent axiom of quantum mechanics, but as a geometric consequence of admissible scalar-time coherence sectors.

7 Discussion

The purpose of this paper has been narrowly defined from the outset. We have not attempted to derive all of quantum mechanics from Time–Scalar Field Theory (TSFT), nor have we attempted to construct a local hidden-variable alternative to Bell correlations. Instead, the objective has been to determine whether the specific geometric structure responsible for two-state quantum correlations can emerge naturally from scalar-time spectral closure.

The resulting derivation suggests that this is possible under a restricted but mathematically coherent set of assumptions.

7.1 What Has Been Derived

Beginning from the scalar-time field

$$\Theta(x^\mu),$$

we constructed the spatial closure operator

$$L_\Theta = -\nabla^2 + V''(\Theta_0),$$

obtained from the scalar-time fluctuation dynamics after spatial-slice spectral reduction.

We then imposed:

1. finite-energy admissibility,
2. normalizability,
3. and scalar-time phase-return closure.

These conditions produced:

1. a discrete admissible spectral sector,
2. orthogonal closure eigenmodes,
3. degenerate two-state closure subspaces,
4. closure-preserving unitary transformations,
5. emergent SU(2)-covariant geometry,
6. projector-valued binary observables,
7. Born-type probability weights,
8. and the Bell correlation law

$$E(a, b) = -a \cdot b.$$

The resulting CHSH bound

$$S_{\max} = 2\sqrt{2}$$

then followed as a direct geometric consequence of the admissible closure manifold.

Importantly, the derivation did not begin by postulating:

1. Hilbert-space quantum mechanics,
2. spin operators,
3. Bell-state geometry,
4. or Born-rule probabilities.

Instead, these structures emerged sequentially from:

1. self-adjoint scalar-time closure spectra,
2. recurrence-compatible admissibility,
3. degenerate closure sectors,
4. closure-preserving projective geometry,
5. and rotational covariance on the emergent two-state manifold.

7.2 Relationship to Standard Quantum Mechanics

The present work does not invalidate ordinary quantum mechanics. Rather, it proposes a possible deeper structural origin for portions of its two-state geometry.

In conventional quantum theory:

1. Hilbert-space structure is assumed,
2. unitary symmetry is assumed,
3. and projector-valued measurements are assumed.

Within the present framework, these structures appear as consequences of admissible scalar-time closure conditions.

The resulting equivalence is therefore structural rather than interpretive. We do not claim that scalar time replaces quantum mechanics operationally. Instead, the claim is that quantum two-state geometry may arise naturally from deeper closure-preserving spectral constraints.

7.3 Why Bell Violations Occur

The present derivation also clarifies why Bell inequalities fail.

Classical Bell inequalities rely on:

1. Kolmogorov probability geometry,
2. simultaneously well-defined hidden observables,
3. and factorized local measurement structure.

The closure framework derived here possesses none of these features fundamentally.

Instead:

1. admissible observables are projector-valued,
2. measurement structure depends on closure orientation,
3. and binary outcomes arise from rotational geometry on the SU(2) manifold.

Bell violations therefore occur not because "local realism fails" in a vague philosophical sense, but because the admissible closure geometry is nonclassical from the outset.

The experimentally observed Tsirelson bound is then the maximal correlation compatible with this geometry.

7.4 Limitations

Several important limitations should be emphasized.

First, the present derivation addresses only finite-dimensional two-state closure sectors. Extension to:

1. higher-dimensional sectors,
2. continuous spectra,
3. quantum field structure,
4. and interacting many-body systems

remains unresolved.

Second, while the derivation avoids directly postulating Hilbert-space quantum mechanics, it still relies on:

1. self-adjoint spectral operators,
2. inner-product-preserving transformations,
3. and closure-compatible normalization.

These assumptions are physically motivated by closure preservation, but their ultimate dynamical origin within the full TSFT framework requires further analysis.

Third, the present work addresses only measurement geometry and correlation structure. It does not yet derive:

1. decoherence dynamics,
2. relativistic quantum field structure,
3. gauge interactions,
4. or the Standard Model particle hierarchy.

Thus the present paper should be interpreted as a geometric closure result rather than a complete quantum reconstruction.

7.5 Implications for the TSFT Program

Despite these limitations, the derivation establishes an important milestone within the broader TSFT framework.

Earlier TSFT work established:

1. scalar-time dynamics,
2. spectral closure structure,
3. uncertainty geometry,
4. and emergent spinor behavior.

The present paper extends this progression by demonstrating that Bell-type quantum correlations emerge naturally once closure-preserving degenerate sectors are analyzed geometrically.

This result strengthens the broader TSFT hypothesis that quantum structure may reflect admissible coherence geometry within an underlying scalar-time manifold rather than irreducible probabilistic axioms.

7.6 Future Directions

Several directions for further development are immediate.

First, the present derivation should be extended to:

1. higher SU(N) closure sectors,
2. multipartite Bell structures,
3. and generalized entanglement geometry.

Second, it may be possible to derive relativistic spinor field structure directly from interacting closure sectors rather than treating two-state geometry independently.

Third, the relationship between closure geometry and decoherence may provide a pathway toward understanding apparent wavefunction collapse dynamically within scalar-time evolution.

Finally, experimental distinctions between ordinary quantum mechanics and scalar-time closure geometry remain unknown. Determining whether the present framework produces genuinely testable deviations represents one of the most important open questions for future investigation.

7.7 Interpretation of Closure

An important interpretive question underlying the present framework is the meaning of the term "closure" itself.

Within the mathematical development of this paper, closure has been defined operationally through:

1. recurrence under scalar-time evolution,
2. finite-energy admissibility,
3. normalizability,
4. and preservation of coherent phase structure.

However, the broader physical interpretation motivating these conditions is that the scalar-time field carries propagating informational structure. In this view, admissible physical states correspond to stable informational coherence configurations capable of persisting under scalar-time evolution.

The closure conditions imposed throughout the paper therefore select configurations whose informational structure remains globally self-consistent during propagation. Continuous spectral sectors generically dephase and disperse, while recurrent point-spectrum sectors preserve coherent informational organization over extended scalar-time evolution.

From this perspective:

1. spectral eigenmodes correspond to stable coherence channels,
2. recurrence corresponds to informational persistence,
3. interference reflects overlap geometry between admissible coherence structures,

4. and quantization emerges because only certain recurrent closure configurations remain globally stable.

Importantly, the present paper does not require a fully developed metaphysical interpretation of information, consciousness, or ontology. The role of information here is operational and dynamical rather than philosophical. The central mathematical results depend only on the existence of closure-preserving recurrent coherence sectors generated by the scalar-time field.

Accordingly, the Bell correlations derived in this work are interpreted not as evidence for hidden-variable realism, but as consequences of the projective geometry governing admissible informational closure structures within recurrent scalar-time evolution.

8 Conclusion

This paper has investigated whether the geometric structure responsible for Bell correlations can emerge from scalar-time spectral closure within Time–Scalar Field Theory.

Beginning from the scalar-time field $\Theta(x^\mu)$, we derived a self-adjoint fluctuation operator governing admissible coherence modes. Imposing finite-energy, normalizability, and phase-return conditions produced a discrete closure spectrum. Degenerate admissible eigenspaces were then shown to possess closure-preserving unitary rotational structure. Modulo global phase redundancy, the resulting geometry became SU(2)-covariant.

Within this emergent geometry, binary measurements arose naturally as projector-valued closure decompositions. The corresponding probability structure reduced to squared projection amplitudes, yielding the standard Born form. Correlated two-sector closure states then produced the Bell correlation law

$$E(a, b) = -a \cdot b,$$

and consequently the CHSH Tsirelson bound

$$S_{\max} = 2\sqrt{2}.$$

The central result is therefore not that Bell correlations were inserted axiomatically into the framework, but that they emerged from the geometry of admissible scalar-time closure sectors under the closure and covariance assumptions developed in this work.

This suggests that at least part of quantum two-state structure may reflect deeper closure-preserving spectral constraints rather than irreducible probabilistic postulates. Whether this perspective can ultimately be extended into a broader derivation of quantum field structure remains an open question. However, the present analysis demonstrates that Bell-type measurement geometry can arise naturally within the scalar-time spectral framework without introducing classical hidden-variable factorization or postulating quantum Hilbert-space structure from the outset.

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A Appendix A: Self-Adjoint Spatial Closure Operators and Spectral Stability

This appendix establishes the operator-theoretic foundation underlying the scalar-time closure construction used throughout the paper.

The key point is that the spectral closure analysis is performed not on the full Lorentzian spacetime operator

$$-\Box + V''(\Theta_0),$$

which is hyperbolic, but rather on the spatial closure operator obtained after separation of scalar-time evolution from spatial mode structure.

The resulting operator is elliptic and admits the standard self-adjoint spectral framework required for unitary closure evolution.

A.1 Linearized Scalar-Time Fluctuations

Let

$$\Theta(x^\mu)$$

be a scalar-time field satisfying

$$\Box\Theta = V'(\Theta).$$

Suppose

$$\Theta_0(x)$$

is a stable background solution satisfying

$$\Box\Theta_0 = V'(\Theta_0).$$

Introduce a small fluctuation

$$\Theta(x,t) = \Theta_0(x) + \psi(x,t), \qquad |\psi| \ll 1.$$

Expanding the scalar potential gives

$$V(\Theta_0 + \psi) = V(\Theta_0) + V'(\Theta_0)\psi + \frac{1}{2}V''(\Theta_0)\psi^2 + O(\psi^3).$$

Because the background satisfies the field equation, the linear term cancels. The quadratic fluctuation action becomes

$$S^{(2)}[\psi] = \frac{1}{2}\int d^4x \left[\partial_\mu\psi\,\partial^\mu\psi - V''(\Theta_0)\psi^2\right].$$

Variation yields the linearized fluctuation equation

$$\Box\psi - V''(\Theta_0)\psi = 0.$$

Equivalently,

$$\left(-\Box + V''(\Theta_0)\right)\psi = 0.$$

This equation determines the Lorentz-compatible fluctuation dynamics of the scalar-time field.

A.2 Spatial-Slice Spectral Reduction

The spectral closure analysis used throughout this paper is not performed directly on the Lorentzian operator

$$-\Box + V''(\Theta_0),$$

since that operator is hyperbolic in Lorentzian signature.

Instead, closure spectra are defined on a spatial Cauchy slice

$$\Sigma_t,$$

after separating scalar-time evolution from spatial mode structure.

For stationary or quasistationary backgrounds, write

$$\psi(x, \tau) = u(x)e^{-i\omega\tau},$$

where

$$x \in \Sigma_t$$

and

$$\tau$$

is the scalar-time evolution parameter.

The spatial closure modes satisfy

$$L_\Theta u = \lambda u,$$

where

$$L_\Theta = -\nabla^2 + V''(\Theta_0).$$

The operator acts on

$$L^2(\Sigma_t, d^3x),$$

with inner product

$$\langle u, v\rangle = \int_{\Sigma_t} d^3x\, u^*(x)v(x).$$

Thus the spectral closure problem is elliptic and belongs to the standard Schrödinger-type operator class.

A.3 Symmetry of the Closure Operator

Let

$$u, v \in D(L_\Theta),$$

where

$$D(L_\Theta) \subset L^2(\Sigma_t)$$

is a dense domain of admissible functions satisfying appropriate regularity and boundary conditions.

Then

$$\langle u, L_\Theta v\rangle = \int_{\Sigma_t} d^3x\, u^* \left(-\nabla^2 v + V''(\Theta_0)v\right).$$

Integrating the Laplacian term by parts gives

$$\int_{\Sigma_t} d^3x\, u^*(-\nabla^2 v) = \int_{\Sigma_t} d^3x\, (\nabla u^*)\cdot(\nabla v),$$

assuming boundary terms vanish sufficiently rapidly at spatial infinity or on the boundary of

$$\Sigma_t.$$

Since

$$V''(\Theta_0)$$

is real-valued,

$$\int_{\Sigma_t} d^3x\, u^* V''(\Theta_0) v = \int_{\Sigma_t} d^3x\, \left(V''(\Theta_0)u\right)^* v.$$

Therefore

$$\langle u, L_\Theta v\rangle = \langle L_\Theta u, v\rangle.$$

Thus

$$L_\Theta$$

is symmetric.

A.4 Self-Adjoint Realization

Under the standard regularity and lower-boundedness assumptions on

$$V''(\Theta_0),$$

the operator

$$L_\Theta = -\nabla^2 + V''(\Theta_0)$$

belongs to the standard Schrödinger-type class admitting self-adjoint realizations on

$$L^2(\Sigma_t).$$

In particular:

1. $$-\nabla^2$$

   is essentially self-adjoint on smooth compactly supported functions;

2. multiplication by sufficiently regular real-valued

   $$V''(\Theta_0)$$

preserves self-adjointness under the usual Kato–Rellich conditions;

3. therefore

$$L_\Theta$$

admits a self-adjoint realization generating unitary scalar-time evolution.

Consequently, Stone's theorem implies that

$$U(\tau) = e^{-iL_\Theta \tau}$$

forms a strongly continuous one-parameter unitary group on

$$L^2(\Sigma_t).$$

A.5 Spectral Decomposition

Since

$$L_\Theta$$

is self-adjoint, the spectral theorem applies.

Admissible closure states therefore admit spectral decomposition into generalized eigenmodes:

$$u = \sum_n c_n u_n + \int d\mu(\lambda)\, c(\lambda) u_\lambda.$$

Discrete spectral sectors correspond to recurrent closure modes, while absolutely continuous spectral sectors generically exhibit dispersive dephasing under scalar-time evolution.

The closure framework developed in the main text therefore selects physically admissible recurrent sectors from the self-adjoint spectral structure of

$$L_\Theta.$$

A.6 Unitary Closure Evolution

For every admissible closure state

$$u \in L^2(\Sigma_t),$$

scalar-time evolution is given by

$$u(\tau) = U(\tau)u(0) = e^{-iL_\Theta \tau} u(0).$$

Since

$$U^\dagger(\tau)U(\tau) = I,$$

the evolution preserves:

1. normalization,

2. orthogonality,
3. closure amplitudes,
4. and spectral decomposition.

Therefore the scalar-time closure framework possesses intrinsically unitary recurrence-compatible spectral evolution.

A.7 Physical Interpretation

The essential point of the present appendix is that the closure geometry used throughout the Bell-sector derivation emerges from a well-defined self-adjoint spatial spectral problem.

Lorentz covariance remains encoded in the underlying scalar-time field equation

$$\Box\Theta = V'(\Theta),$$

while admissible closure sectors arise from the self-adjoint spectral decomposition of the associated spatial closure operator

$$L_\Theta = -\nabla^2 + V''(\Theta_0).$$

Thus the recurrence-compatible unitary geometry developed in the main text follows from the spectral structure of admissible scalar-time closure modes.

B Appendix B: Degenerate Closure Sectors and Emergent Unitary Geometry

This appendix derives the internal transformation geometry of degenerate scalar-time closure sectors in full detail.

The central result is that closure-preserving indistinguishability forces the admissible internal transformation group to be unitary. Physical states then arise not as vectors themselves, but as projective rays modulo global phase redundancy. The resulting two-state closure manifold becomes the complex projective line

$$\mathbb{CP}^1 \cong S^2,$$

with

$$SU(2)$$

acting transitively as the spinorial closure symmetry group.

B.1 Degenerate Closure Sector

Suppose

$$L_\Theta\psi_1 = \lambda\psi_1, \qquad L_\Theta\psi_2 = \lambda\psi_2,$$

with orthonormality

$$\langle \psi_i, \psi_j \rangle = \delta_{ij}.$$

The degenerate closure sector is therefore

$$H_\lambda = \text{span}\{\psi_1, \psi_2\}.$$

For arbitrary coefficients

$$c_1, c_2 \in \mathbb{C},$$

the state

$$\Psi = c_1\psi_1 + c_2\psi_2$$

satisfies

$$L_\Theta \Psi = \lambda \Psi.$$

Thus the entire degenerate sector evolves coherently under scalar-time propagation:

$$U(\tau)\Psi = e^{-i\lambda\tau}\Psi.$$

Relative structure inside the degenerate sector is therefore preserved under scalar-time evolution.

B.2 Closure-Preserving Basis Transformations

Consider a transformed basis

$$\psi_i' = \sum_j U_{ij}\psi_j.$$

We now determine which transformations preserve physically admissible closure structure.

The admissibility conditions derived previously require preservation of:

1. closure norm,
2. orthogonality,
3. relative closure amplitudes,
4. and recurrence-compatible phase structure.

For arbitrary states

$$\Psi = \sum_i c_i\psi_i, \qquad \Phi = \sum_i d_i\psi_i,$$

closure-preserving indistinguishability requires

$$\langle \Psi', \Phi' \rangle = \langle \Psi, \Phi \rangle.$$

Writing the transformed states as

$$\Psi' = U\Psi, \qquad \Phi' = U\Phi,$$

gives

$$\langle \Psi', \Phi' \rangle = \Psi^\dagger U^\dagger U \Phi.$$

Since the inner product must be preserved for arbitrary admissible states,

$$U^\dagger U = I.$$

Therefore admissible closure-preserving transformations are unitary.

Proposition B.1. The internal transformation group acting on a degenerate scalar-time closure sector is unitary.

This conclusion follows directly from:

1. spectral degeneracy,
2. closure-preserving indistinguishability,
3. and preservation of closure norm.

No independent Hilbert-space axiom has been introduced.

B.3 Normalized Closure States

Let

$$\Psi = c_1\psi_1 + c_2\psi_2$$

be normalized:

$$|c_1|^2 + |c_2|^2 = 1.$$

The set of normalized closure amplitudes therefore forms the unit sphere

$$S^3 = \left\{(c_1, c_2) \in \mathbb{C}^2 : |c_1|^2 + |c_2|^2 = 1\right\}.$$

Thus the admissible closure amplitudes possess an intrinsic rotational geometry inherited from the unitary structure of the degenerate sector.

B.4 Global Phase Redundancy

Physical closure observables are insensitive to overall global phase.

Indeed,

$$\Psi \mapsto e^{i\alpha}\Psi$$

leaves invariant:

$$|\Psi|^2, \qquad \langle \Psi, \Psi \rangle, \qquad |\langle \Phi, \Psi \rangle|^2,$$

and all relative closure probabilities.

Therefore physically distinguishable states correspond not to vectors themselves, but to equivalence classes:

$$[\Psi] = \{e^{i\alpha}\Psi : \alpha \in \mathbb{R}\}.$$

The physical closure manifold is therefore

$$S^3/U(1).$$

This quotient is precisely the complex projective line:

$$S^3/U(1) = \mathbb{CP}^1.$$

B.5 Emergent Projective Geometry

The complex projective line possesses the well-known geometric identification

$$\mathbb{CP}^1 \cong S^2.$$

Thus normalized two-state closure rays form a sphere-valued projective manifold.

Equivalently, every physical closure state may be parameterized by Bloch coordinates:

$$\Psi(\theta, \phi) = \cos\frac{\theta}{2}\,\psi_1 + e^{i\phi}\sin\frac{\theta}{2}\,\psi_2.$$

The parameters

$$(\theta, \phi)$$

therefore coordinatize the closure sphere.

This is precisely the geometric structure underlying ordinary two-state quantum systems.

B.6 SU(2) Closure Symmetry

The group

$$SU(2)$$

acts naturally on the normalized closure amplitudes:

$$\Psi \mapsto U\Psi, \qquad U \in SU(2).$$

Since

$$U^\dagger U = I,$$

the action preserves:

1. normalization,
2. inner products,
3. closure amplitudes,
4. and projective geometry.

The induced action on projective rays corresponds to rotations of the closure sphere.

Thus:

$$SU(2)$$

acts transitively on

$$\mathbb{CP}^1 \cong S^2.$$

The resulting geometry is therefore:

1. spinorial at the amplitude level,
2. and spherical at the projective physical-state level.

B.7 Physical Interpretation

The emergence of this geometry has a direct scalar-time closure interpretation.

Within a degenerate closure sector:

1. admissible coherence modes cannot be physically distinguished by internal unitary rotations preserving closure norm;
2. physically meaningful observables depend only on relative closure orientation;
3. and global phase possesses no observable significance.

Consequently:

$$\text{degenerate closure amplitudes} + \text{unitary preservation} + \text{global phase redundancy}$$

produce:

$$\mathbb{CP}^1 \cong S^2,$$

with

$$SU(2)$$

as the natural closure symmetry group.

Thus the projective geometry underlying two-state quantum structure emerges from closure-preserving scalar-time spectral organization rather than from an independently postulated Hilbert-space axiom.

C Appendix C: Emergence of the Bloch Sphere

Normalized closure states satisfy

$$|c_1|^2 + |c_2|^2 = 1.$$

Parameterize:

$$c_1 = e^{i\gamma} \cos\frac{\theta}{2},$$

$$c_2 = e^{i(\gamma+\phi)} \sin \frac{\theta}{2}.$$

Factoring out global phase:

$$\Psi(\theta, \phi) = \cos \frac{\theta}{2} \psi_1 + e^{i\phi} \sin \frac{\theta}{2} \psi_2.$$

Thus physical closure states are parameterized by:

$$(\theta, \phi) \in S^2.$$

The closure manifold is therefore geometrically identical to the Bloch sphere.

D Appendix D: Closure Covariance and Probability Compatibility

This appendix establishes the compatibility of quadratic closure probabilities with the symmetry structure of admissible scalar-time closure sectors.

The goal of the present appendix is intentionally limited. We do not attempt here to derive the full probability geometry uniquely from covariance assumptions alone. Rather, we show that quadratic overlap probabilities are naturally compatible with:

1. closure-preserving unitary transformations,
2. global-phase invariance,
3. and the projective geometry of admissible two-state closure sectors.

The stronger geometric derivation of the quadratic probability structure from the Fubini–Study metric on the projective closure manifold is developed separately in Appendix I.

D.1 Closure Amplitudes

Let

$$H_\lambda = \text{span}\{\psi_1, \psi_2\}$$

be a degenerate admissible closure sector.

A normalized closure state takes the form

$$\Psi = c_1\psi_1 + c_2\psi_2,$$

with

$$|c_1|^2 + |c_2|^2 = 1.$$

Given another admissible closure state

$$\Phi,$$

the overlap amplitude is

$$A(\Phi, \Psi) = \langle \Phi, \Psi \rangle.$$

Physical transition probabilities must therefore depend on closure overlaps between admissible projective states.

D.2 Global-Phase Invariance

Physical closure observables are insensitive to overall phase.

Under the transformation

$$\Psi \mapsto e^{i\alpha} \Psi,$$

the overlap amplitude transforms as

$$A(\Phi, \Psi) \mapsto e^{i\alpha} A(\Phi, \Psi).$$

Therefore physically admissible probabilities cannot depend on the complex phase of

$$A(\Phi, \Psi)$$

itself.

Instead, admissible probabilities must depend only on phase-invariant quantities such as

$$|A(\Phi, \Psi)|.$$

D.3 Closure-Covariant Transformations

Let

$$U \in SU(2)$$

be a closure-preserving unitary transformation acting on the degenerate closure sector.

Then

$$\Psi \mapsto U\Psi, \qquad \Phi \mapsto U\Phi.$$

Since

$$U^\dagger U = I,$$

the overlap amplitude transforms as

$$\langle U\Phi, U\Psi \rangle = \langle \Phi, \Psi \rangle.$$

Hence the modulus

$$|\langle \Phi, \Psi \rangle|$$

is invariant under admissible closure rotations.

Therefore any physically admissible probability assignment compatible with closure covariance must depend only on the invariant projective overlap structure.

D.4 Quadratic Compatibility

The quadratic overlap expression

$$P(\Phi|\Psi) = |\langle \Phi, \Psi \rangle|^2$$

satisfies:

1. positivity,
2. normalization,
3. global-phase invariance,
4. and closure covariance under SU(2) transformations.

Indeed:

$$P(U\Phi|U\Psi) = |\langle U\Phi, U\Psi \rangle|^2 = |\langle \Phi, \Psi \rangle|^2.$$

Thus quadratic overlap probabilities are fully compatible with the closure symmetry structure of the admissible projective manifold.

D.5 Relation to Projective Geometry

The admissible physical closure states form the projective manifold

$$\mathbb{CP}^1 \cong S^2.$$

On this manifold, physically meaningful distinctions depend only on relative projective orientation and not on arbitrary overall phase conventions.

Consequently, transition probabilities must be functions of projective overlap geometry.

The detailed geometric derivation showing that the unique continuous SU(2)-covariant transition probability on the two-state closure manifold takes the quadratic form

$$P(\Phi|\Psi) = |\langle \Phi, \Psi \rangle|^2$$

is developed in Appendix I using the Fubini–Study metric.

D.6 Physical Interpretation

The present appendix establishes that quadratic overlap probabilities are naturally compatible with:

1. closure-preserving unitary geometry,
2. projective state reduction,
3. and global-phase redundancy.

Thus the probability structure appearing in the Bell-sector derivation is consistent with the symmetry structure of admissible scalar-time closure sectors.

The stronger geometric reconstruction of the quadratic probability law itself is deferred to Appendix I.

E Appendix E: Detailed Bell Correlation Derivation

E.1 Singlet Closure State

Define:

$$\Psi_S = \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_- - \psi_- \otimes \psi_+).$$

E.2 Pauli Algebra

The Pauli matrices satisfy:

$$\sigma_i \sigma_j = \delta_{ij} I + i\epsilon_{ijk}\sigma_k.$$

Using antisymmetry of Ψ_S,

$$\langle \Psi_S, \sigma_i \otimes \sigma_j \Psi_S \rangle = -\delta_{ij}.$$

E.3 Correlation Function

Define:

$$\Sigma(a) = a \cdot \boldsymbol{\sigma}.$$

Then:

$$E(a,b) = \langle \Psi_S, \Sigma(a) \otimes \Sigma(b) \Psi_S \rangle.$$

Expanding:

$$E(a,b) = \sum_{ij} a_i b_j \langle \Psi_S, \sigma_i \otimes \sigma_j \Psi_S \rangle.$$

Using:

$$\langle \sigma_i \otimes \sigma_j \rangle = -\delta_{ij},$$

we obtain:

$$E(a,b) = -\sum_i a_i b_i.$$

Hence:

$$E(a,b) = -a \cdot b.$$

F Appendix F: Operator Derivation of the Tsirelson Bound

This appendix derives the maximal Bell correlation compatible with the admissible two-state closure geometry developed in the main text.

The derivation follows entirely from:

1. closure-preserving SU(2)-covariant observables,
2. the Pauli operator algebra,
3. and the projective geometry of the admissible closure manifold.

F.1 Binary Closure Observables

Let
$$a, a', b, b' \in \mathbb{R}^3$$
be unit vectors specifying admissible closure measurement directions.

Define the corresponding binary observables
$$A = a \cdot \sigma, \qquad A' = a' \cdot \sigma, \qquad B = b \cdot \sigma, \qquad B' = b' \cdot \sigma,$$
where
$$\sigma = (\sigma_x, \sigma_y, \sigma_z)$$
denotes the Pauli vector.

Since the Pauli matrices satisfy
$$\sigma_i^2 = I,$$
it follows that
$$A^2 = (A')^2 = B^2 = (B')^2 = I.$$

Thus each observable possesses eigenvalues
$$\pm 1.$$

F.2 CHSH Operator

Define the CHSH operator
$$\mathcal{B} = A \otimes (B + B') + A' \otimes (B - B').$$

Equivalently,
$$\mathcal{B} = A \otimes B + A \otimes B' + A' \otimes B - A' \otimes B'.$$

The maximal Bell correlation is determined by the operator norm of
$$\mathcal{B}.$$

F.3 Expansion of the CHSH Operator Square

Expanding the square gives
$$\mathcal{B}^2 = A^2 \otimes (B + B')^2 + (A')^2 \otimes (B - B')^2 + AA' \otimes (B + B')(B - B') + A'A \otimes (B - B')(B + B').$$

Using

$$A^2 = (A')^2 = B^2 = (B')^2 = I,$$

we compute

$$(B + B')^2 = 2I + \{B, B'\},$$

and

$$(B - B')^2 = 2I - \{B, B'\},$$

where

$$\{X, Y\} = XY + YX$$

denotes the anticommutator.

Therefore the first two terms combine to give

$$4I.$$

Now compute the mixed terms:

$$(B + B')(B - B') = -[B, B'],$$

and

$$(B - B')(B + B') = [B, B'].$$

Thus

$$\mathcal{B}^2 = 4I - [A, A'] \otimes [B, B'].$$

F.4 Pauli Commutator Structure

The Pauli algebra satisfies

$$[a \cdot \sigma, a' \cdot \sigma] = 2i(a \times a') \cdot \sigma.$$

Similarly,

$$[b \cdot \sigma, b' \cdot \sigma] = 2i(b \times b') \cdot \sigma.$$

Hence

$$[A, A'] \otimes [B, B'] = (2i)^2\big((a \times a') \cdot \sigma\big) \otimes \big((b \times b') \cdot \sigma\big).$$

Since

$$(2i)^2 = -4,$$

we obtain

$$\mathcal{B}^2 = 4I + 4\big((a \times a') \cdot \sigma\big) \otimes \big((b \times b') \cdot \sigma\big).$$

F.5 Operator Norm Estimate

For any vector

$$v \in \mathbb{R}^3,$$

the Pauli operator satisfies

$$(v \cdot \sigma)^2 = |v|^2 I.$$

Therefore

$$\|v \cdot \sigma\| = |v|.$$

Consequently,

$$\left\|(a \times a') \cdot \sigma\right\| = |a \times a'| \leq 1,$$

and similarly

$$\left\|(b \times b') \cdot \sigma\right\| = |b \times b'| \leq 1.$$

Using submultiplicativity of the operator norm,

$$\|\mathcal{B}^2\| \leq 4 + 4 \left\|(a \times a') \cdot \sigma\right\| \left\|(b \times b') \cdot \sigma\right\|.$$

Hence

$$\|\mathcal{B}^2\| \leq 4 + 4 = 8.$$

Therefore

$$\|\mathcal{B}\| \leq 2\sqrt{2}.$$

F.6 Tsirelson Bound

Let

$$\Psi$$

be any normalized admissible two-sector closure state.

Then

$$|\langle \Psi, \mathcal{B}\Psi \rangle| \leq \|\mathcal{B}\| \leq 2\sqrt{2}.$$

Therefore the CHSH functional satisfies

$$|S| \leq 2\sqrt{2}.$$

Thus the maximal Bell correlation compatible with the admissible scalar-time closure geometry is

$$S_{\max} = 2\sqrt{2}.$$

F.7 Physical Interpretation

The Tsirelson bound therefore follows directly from:

1. closure-preserving SU(2) geometry,
2. projective two-state structure,
3. and the Pauli commutator algebra.

No hidden-variable assumption or superluminal signaling mechanism enters the derivation.

Instead, the Bell correlation structure emerges from the geometry of admissible scalar-time closure sectors themselves.

G Appendix G: Recurrence, Spectral Closure, and Admissible Scalar-Time Sectors

This appendix formalizes the recurrence structure underlying admissible scalar-time closure modes.

The central goal is to distinguish recurrent closure sectors from dispersive sectors under unitary scalar-time evolution generated by the self-adjoint spatial closure operator

$$L_\Theta = -\nabla^2 + V''(\Theta_0).$$

The admissibility framework developed here provides the spectral basis for the closure-selection principle used throughout the main text.

G.1 Spatial Closure Evolution

Let

$$L_\Theta = -\nabla^2 + V''(\Theta_0)$$

act on

$$L^2(\Sigma_t, d^3x),$$

where

$$\Sigma_t$$

is a spatial Cauchy slice.

Assume

$$L_\Theta$$

admits a self-adjoint realization.

By Stone's theorem, scalar-time evolution is generated by the strongly continuous unitary group

$$U(\tau) = e^{-iL_\Theta \tau}.$$

For an admissible closure state

$$u \in L^2(\Sigma_t),$$

scalar-time evolution therefore takes the form

$$u(\tau) = U(\tau)u(0).$$

Because

$$U^\dagger(\tau)U(\tau) = I,$$

closure norm is preserved:

$$\|u(\tau)\| = \|u(0)\|.$$

G.2 Recurrence-Compatible Closure States

The closure framework selects recurrent scalar-time coherence structures.

A closure state

$$u \in L^2(\Sigma_t)$$

is called recurrence-compatible if there exists a sequence

$$\tau_n \to \infty$$

and phases

$$\phi_n \in \mathbb{R}$$

such that

$$\left\| U(\tau_n)u - e^{i\phi_n}u \right\| \to 0.$$

Thus admissible closure sectors are not required to exhibit exact finite-period recurrence. Instead, admissibility requires asymptotic phase-return compatibility under scalar-time evolution.

This is the appropriate almost-periodic recurrence condition for the spectral closure framework.

G.3 Spectral Decomposition

Since

$$L_\Theta$$

is self-adjoint, the spectral theorem applies.

Every closure state therefore admits decomposition into spectral components:

$$u = \sum_n c_n u_n + \int d\mu(\lambda)\, c(\lambda) u_\lambda.$$

Here:

1. $$u_n$$

   belong to the discrete spectrum;

2. $$u_\lambda$$

   belong to the continuous spectrum;

3. $$d\mu(\lambda)$$

is the spectral measure associated with

$$L_\Theta.$$

Under scalar-time evolution,

$$u(\tau) = \sum_n c_n e^{-i\lambda_n \tau} u_n + \int d\mu(\lambda)\, c(\lambda) e^{-i\lambda\tau} u_\lambda.$$

Thus recurrence properties depend directly on spectral structure.

G.4 Discrete Spectral Recurrence

Suppose

$$u = \sum_{n=1}^{N} c_n u_n$$

belongs entirely to a finite discrete spectral sector.

Then scalar-time evolution gives

$$u(\tau) = \sum_{n=1}^{N} c_n e^{-i\lambda_n \tau} u_n.$$

The overlap amplitude becomes

$$A(\tau) = \langle u, u(\tau) \rangle = \sum_{n=1}^{N} |c_n|^2 e^{-i\lambda_n \tau}.$$

Because the evolution is a finite superposition of phase rotations, the trajectory is almost-periodic in the sense of Bohr.

Consequently, for sufficiently compatible rational approximations among the frequencies

$$\lambda_n,$$

there exist sequences

$$\tau_n$$

such that

$$\left\| u(\tau_n) - e^{i\phi_n} u \right\| \to 0.$$

Therefore finite discrete spectral sectors satisfy recurrence-compatible closure evolution.

G.5 Singular Continuous Spectra

The preceding conclusion does not automatically extend to singular continuous spectral measures.

Certain singular continuous spectra may support nontrivial recurrent or quasirecurrent behavior depending on the detailed structure of the spectral measure.

The present closure framework therefore does not claim that all non-discrete spectra are excluded purely by abstract spectral theory alone.

Instead, the admissibility principle adopted throughout this paper selects recurrence-compatible closure sectors under the regularity assumptions relevant to physically stable scalar-time coherence structures.

In the applications considered here, the physically relevant admissible sectors are discrete or effectively discrete recurrent closure spectra.

G.6 Closure Admissibility Principle

The spectral closure framework therefore adopts the following admissibility principle:

Closure Admissibility Principle. Physically admissible scalar-time closure sectors are those possessing recurrence-compatible phase evolution under the unitary scalar-time flow generated by

$$L_\Theta.$$

This principle excludes:

1. dispersive dephasing sectors,
2. irreversibly incoherent spectral sectors,
3. and unstable nonrecurrent closure configurations.

Conversely, recurrent closure sectors support:

1. persistent coherence,
2. stable phase structure,
3. and closure-preserving observables.

G.7 Physical Interpretation

The central physical idea is that scalar-time closure selects coherence-stable recurrent spectral structures.

Discrete recurrent sectors repeatedly realign their scalar-time phase structure under unitary evolution. These sectors therefore support stable closure observables and persistent coherence relations.

By contrast, dispersive absolutely continuous sectors generically dephase under scalar-time evolution and fail to preserve coherent closure structure.

Thus admissible scalar-time sectors arise not merely from formal spectral existence, but from recurrence-compatible closure stability under scalar-time evolution.

H Appendix H: Exact Geometry of Two-State Closure Sectors

This appendix corrects and sharpens the geometric statement used in the main text.

The physically relevant conclusion is not the loose identification

$$U(2)/U(1) \cong SU(2),$$

which is not the correct projective-state statement. The precise statement is:

$$\mathbb{CP}^1 \cong S^2,$$

and $SU(2)$ acts transitively on this space, with a $U(1)$ stabilizer.

Thus the emergent geometry of a normalized two-state scalar-time closure sector is the complex projective line, equivalently the Bloch sphere.

H.1 Two-Dimensional Closure Sector

Let

$$\mathcal{H}_\lambda = \text{span}\{\psi_1, \psi_2\}$$

be a two-dimensional degenerate scalar-time closure sector satisfying

$$L_\Theta \psi_1 = \lambda \psi_1, \qquad L_\Theta \psi_2 = \lambda \psi_2,$$

with

$$\langle \psi_i, \psi_j \rangle = \delta_{ij}.$$

A general normalized state is

$$\Psi = c_1 \psi_1 + c_2 \psi_2,$$

where

$$(c_1, c_2) \in \mathbb{C}^2, \qquad |c_1|^2 + |c_2|^2 = 1.$$

Thus normalized states form the unit 3-sphere:

$$S^3 = \{(c_1, c_2) \in \mathbb{C}^2 : |c_1|^2 + |c_2|^2 = 1\}.$$

H.2 Global Phase Redundancy

Physical closure amplitudes are invariant under

$$\Psi \mapsto e^{i\alpha} \Psi.$$

Therefore physically distinct states are not points of S^3, but equivalence classes:

$$[\Psi] = \{e^{i\alpha} \Psi : \alpha \in \mathbb{R}\}.$$

The physical state space is therefore

$$S^3/U(1).$$

This quotient is the complex projective line:

$$S^3/U(1) = \mathbb{CP}^1.$$

H.3 Explicit Hopf Map

The identification

$$\mathbb{CP}^1 \cong S^2$$

is realized by the Hopf map.

Define

$$\Psi = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Let

$$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

be the Pauli matrices.

Define the real vector

$$\boldsymbol{n}_\Psi = \Psi^\dagger \boldsymbol{\sigma} \Psi.$$

In components:

$$n_x = c_1^* c_2 + c_2^* c_1,$$

$$n_y = -i(c_1^* c_2 - c_2^* c_1),$$

$$n_z = |c_1|^2 - |c_2|^2.$$

A direct calculation gives

$$|\boldsymbol{n}_\Psi|^2 = 1.$$

Thus:

$$\boldsymbol{n}_\Psi \in S^2.$$

Moreover, if

$$\Psi \mapsto e^{i\alpha}\Psi,$$

then

$$\boldsymbol{n}_{e^{i\alpha}\Psi} = (e^{-i\alpha}\Psi^\dagger)\boldsymbol{\sigma}(e^{i\alpha}\Psi) = \Psi^\dagger \boldsymbol{\sigma} \Psi = \boldsymbol{n}_\Psi.$$

Therefore the Hopf map descends to a well-defined map on projective states:

$$\mathbb{CP}^1 \longrightarrow S^2.$$

This map is bijective and smooth, giving:

$$\mathbb{CP}^1 \cong S^2.$$

H.4 Bloch Parameterization

Choose coordinates:

$$c_1 = e^{i\gamma} \cos \frac{\theta}{2},$$

$$c_2 = e^{i(\gamma+\phi)} \sin \frac{\theta}{2}.$$

Factoring out the global phase $e^{i\gamma}$, every physical state can be written:

$$\Psi(\theta, \phi) = \cos \frac{\theta}{2}\psi_1 + e^{i\phi} \sin \frac{\theta}{2}\psi_2.$$

Then:

$$\boldsymbol{n}_\Psi = (\sin\theta\cos\phi, \, \sin\theta\sin\phi, \, \cos\theta).$$

Thus the physical closure state is equivalently a point on the unit sphere.

H.5 SU(2) Action

The group $SU(2)$ acts on normalized state vectors by:

$$\Psi \mapsto U\Psi, \qquad U \in SU(2).$$

Since

$$U^\dagger U = I,$$

normalization is preserved:

$$\|U\Psi\| = \|\Psi\|.$$

Since U is linear and unitary, inner products and closure amplitudes are preserved.

The corresponding action on Bloch vectors is:

$$\boldsymbol{n}_\Psi \mapsto \boldsymbol{n}_{U\Psi}.$$

For every $U \in SU(2)$, there exists a rotation

$$R_U \in SO(3)$$

such that:

$$\boldsymbol{n}_{U\Psi} = R_U \boldsymbol{n}_\Psi.$$

This is the standard double-cover homomorphism:

$$SU(2) \to SO(3).$$

Thus $SU(2)$ acts as the spinorial lift of ordinary rotations on the closure sphere.

H.6 Stabilizer and Projective Geometry

Fix the reference state

$$\Psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The subgroup of $SU(2)$ preserving the corresponding ray is:

$$\left\{ \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} : \alpha \in \mathbb{R} \right\} \cong U(1).$$

Therefore:

$$\mathbb{CP}^1 \cong SU(2)/U(1).$$

This is the precise geometric quotient relevant to the two-state closure sector.

H.7 Corrected Geometry Statement

The correct result is therefore:

$$\boxed{\text{Normalized two-state closure rays form } \mathbb{CP}^1 \cong S^2, \text{ with } SU(2) \text{ acting transitively.}}$$

This replaces the imprecise statement:

$$U(2)/U(1) \cong SU(2).$$

The distinction matters.

The group $U(2)$ is the full unitary group acting on two-component closure amplitudes. Removing an overall phase gives:

$$U(2)/U(1) \cong PU(2),$$

where $PU(2)$ is the projective unitary group. For two-state systems:

$$PU(2) \cong SO(3).$$

Meanwhile:

$$SU(2)$$

is the double cover of $SO(3)$, and it acts naturally on the spinorial closure amplitudes before projection to rays.

Thus the closure sector contains two related structures:

1. spinorial amplitude geometry governed by $SU(2)$,
2. projective physical-state geometry given by $\mathbb{CP}^1 \cong S^2$.

H.8 Physical Interpretation in TSFT

Within the TSFT closure framework, this means:

1. the two degenerate closure modes form a complex two-component amplitude space;
2. normalization restricts admissible amplitudes to S^3;
3. global phase redundancy projects S^3 to $\mathbb{CP}^1$;
4. $\mathbb{CP}^1$ is geometrically the Bloch sphere;
5. and $SU(2)$ is the closure-preserving spinorial symmetry acting on the amplitudes.

Thus quantum two-state geometry is not inserted as a separate axiom. It follows from:

$$\text{degenerate complex closure amplitudes} + \text{normalization} + \text{global phase redundancy} + \text{unitary closure preservation.}$$

This appendix supplies the exact mathematical geometry needed for the Bell derivation in the main text.

I Appendix I: Closure-Covariant Probability Geometry and the Emergence of Quadratic Amplitude Structure

The purpose of this appendix is to place the probability structure used throughout the main text on a more rigorous geometric footing.

In earlier drafts, quadratic probability weights were motivated through a power-law consistency argument. While that argument correctly identifies the exponent $p = 2$ within the restricted class $P(A) = |A|^p$, it does not uniquely derive the quadratic form itself. The present appendix therefore replaces that restricted argument with a geometric derivation based on the unique SU(2)-invariant projective geometry of the two-state closure manifold.

The goal is intentionally narrow.

We do not claim here to derive the full Born rule in arbitrary infinite-dimensional Hilbert spaces. Rather, we establish the following restricted result:

Claim. Within a normalized two-state scalar-time closure sector, if admissible transition probabilities are required to satisfy:

1. positivity,
2. normalization,
3. continuity,
4. global-phase invariance,
5. and covariance under closure-preserving SU(2) transformations,

then the unique admissible transition probability is determined by the Fubini–Study geometry of the projective closure manifold and takes the quadratic form

$$P(\phi|\Psi) = |\langle \phi, \Psi \rangle|^2.$$

I.1 Two-State Closure Sector

Let

$$H_\lambda = \mathrm{span}\{\psi_1, \psi_2\}$$

be a normalized degenerate scalar-time closure sector.

An admissible normalized closure state is

$$\Psi = c_1\psi_1 + c_2\psi_2,$$

with

$$|c_1|^2 + |c_2|^2 = 1.$$

Physical states correspond not to vectors themselves, but to rays:

$$[\Psi] = \{e^{i\alpha}\Psi : \alpha \in \mathbb{R}\}.$$

Thus the physical closure manifold is

$$\mathbb{CP}^1 \cong S^2,$$

as established in Appendix H.

I.2 Projective Geometry and the Fubini–Study Metric

The complex projective line $\mathbb{CP}^1$ possesses a unique SU(2)-invariant Riemannian metric up to overall scale: the Fubini–Study metric.

For normalized rays $[\Psi]$ and $[\phi]$, the corresponding projective distance is

$$d_{\mathrm{FS}}(\Psi, \phi) = \arccos(|\langle \phi, \Psi \rangle|)\,.$$

Several facts are immediate:

1. $d_{\mathrm{FS}} = 0$ iff the rays coincide.
2. $d_{\mathrm{FS}} = \pi/2$ iff the rays are orthogonal.
3. The metric is invariant under SU(2) closure rotations:

$$d_{\mathrm{FS}}(U\Psi, U\phi) = d_{\mathrm{FS}}(\Psi, \phi).$$

4. The metric depends only on projective geometry and is therefore invariant under global phase.

Thus the physically meaningful distinguishability structure of the two-state closure sector is completely determined by the Fubini–Study geometry.

I.3 Closure Transition Probabilities

We now seek a physically admissible transition probability assignment

$$P(\phi|\Psi),$$

representing the probability that closure state Ψ yields outcome ϕ.

The closure framework requires:

1. positivity:
$$P(\phi|\Psi) \geq 0,$$

2. normalization:
$$P(\Psi|\Psi) = 1,$$

3. orthogonality:
$$P(\phi|\Psi) = 0 \quad \text{if} \quad \langle \phi, \Psi \rangle = 0,$$

4. continuity on the projective manifold,

5. global-phase invariance,

6. and SU(2)-covariance:
$$P(U\phi|U\Psi) = P(\phi|\Psi).$$

Because the probability assignment is SU(2)-covariant and phase-invariant, it cannot depend on arbitrary coordinate choices or basis labels. It may depend only on the invariant projective distance:

$$d_{\text{FS}}(\Psi, \phi).$$

Therefore there exists a continuous function

$$F : [0, \pi/2] \to [0, 1]$$

such that

$$P(\phi|\Psi) = F(d_{\text{FS}}(\Psi, \phi)).$$

Normalization and orthogonality require:

$$F(0) = 1, \qquad F(\pi/2) = 0.$$

I.4 Frame Compatibility and the Fubini–Study Probability Structure

The preceding sections established that admissible transition probabilities on the two-state closure manifold must:

1. depend only on projective separation,
2. remain invariant under SU(2)-covariant closure transformations,
3. and vary continuously with the Fubini–Study distance.

However, continuity and monotonicity alone do not uniquely determine the transition law. An additional compatibility condition is required.

Let

$$\{\phi, \phi^{\perp}\}$$

be any orthonormal closure frame spanning the admissible two-state sector.

Closure probabilities must satisfy frame normalization:

$$P(\phi|\Psi) + P(\phi^{\perp}|\Psi) = 1$$

for every normalized admissible state

$$\Psi.$$

Now let

$$d_{FS}(\Psi, \phi) = \theta.$$

Since the orthogonal complement corresponds to antipodal separation on the Bloch sphere,

$$d_{FS}(\Psi, \phi^{\perp}) = \frac{\pi}{2} - \theta.$$

Define

$$f(\theta) = P(\phi|\Psi).$$

Frame compatibility therefore requires

$$f(\theta) + f\left(\frac{\pi}{2} - \theta\right) = 1.$$

The admissible probability assignment must additionally satisfy:

$$f(0) = 1, \qquad f\left(\frac{\pi}{2}\right) = 0,$$

together with continuity and SU(2)-covariance.

Now recall that the Fubini–Study metric on

$$\mathbb{CP}^1$$

is induced from the round metric on

$$S^2$$

through the Hopf projection

$$S^3 \to \mathbb{CP}^1.$$

The corresponding projective overlap amplitude between normalized closure states is

$$|\langle \phi, \Psi \rangle| = \cos\theta.$$

Therefore the unique frame-compatible quadratic overlap assignment is

$$P(\phi|\Psi) = |\langle \phi, \Psi \rangle|^2 = \cos^2\theta.$$

Indeed,

$$\cos^2\theta + \cos^2\left(\frac{\pi}{2} - \theta\right) = \cos^2\theta + \sin^2\theta = 1,$$

so the frame condition is satisfied identically.

Thus within the admissible two-state closure sector, the unique continuous SU(2)-covariant frame-compatible transition probability induced by the projective overlap geometry is

$$P(\phi|\Psi) = |\langle \phi, \Psi \rangle|^2.$$

Equivalently,

$$P(\phi|\Psi) = \cos^2(d_{FS}(\phi, \Psi)).$$

Therefore the Born-type probability structure follows from:

1. projective closure geometry,
2. frame normalization,
3. SU(2)-covariance,
4. and the Fubini–Study metric structure of the admissible closure manifold.

I.5 Geometric Interpretation

The emergence of quadratic amplitudes is therefore geometric rather than algebraic.

The key point is not merely that amplitudes are complex, but that physically admissible closure states form a projective SU(2)-covariant manifold whose distinguishability structure is governed by the Fubini–Study metric.

Within this geometry:

$$|\langle \phi, \Psi \rangle| = \cos(d_{\mathrm{FS}}),$$

and probability corresponds to the square of the projective overlap cosine.

Thus the quadratic structure arises because transition probability measures geodesic overlap on the closure manifold.

I.6 Consistency with Closure Rotations

Let

$$U \in SU(2)$$

be an admissible closure-preserving transformation.

Since the Fubini–Study metric is SU(2)-invariant,

$$d_{\mathrm{FS}}(U\Psi, U\phi) = d_{\mathrm{FS}}(\Psi, \phi).$$

Therefore

$$P(U\phi|U\Psi) = P(\phi|\Psi).$$

Thus the quadratic probability structure is automatically closure-covariant under admissible rotational transformations.

I.7 Relation to the Main Bell Derivation

The Bell-correlation structure derived in the main text depends only on:

1. the SU(2)-covariant geometry of the two-state closure sector,
2. projector-valued binary observables,
3. and quadratic transition probabilities.

The present appendix shows that the quadratic structure follows directly from the geometry of the projective closure manifold itself rather than from an independently postulated probability axiom.

Consequently, the Bell correlations obtained in the main text arise as geometric consequences of closure-preserving scalar-time coherence structure.

I.8 Scope and Limitations

The derivation presented here is intentionally restricted to finite-dimensional two-state closure sectors.

We do not claim:

1. a general Gleason-type derivation in arbitrary Hilbert spaces,
2. a derivation for infinite-dimensional field sectors,
3. or a complete reconstruction of quantum probability theory.

Rather, the result established here is narrower:

Within an admissible two-state scalar-time closure sector, the unique continuous SU(2)-covariant transition probability compatible with projective closure geometry is quadratic in the overlap amplitude.

This restricted result is sufficient for the Bell-correlation derivation developed in the main text.

I.9 Conclusion

The probability structure used throughout the Bell-sector derivation is therefore not inserted as an independent axiom.

Instead:

$$\text{projective closure geometry} \quad \Longrightarrow \quad \text{Fubini–Study distance} \quad \Longrightarrow \quad \text{quadratic overlap probability.}$$

The resulting probability law

$$P(\phi|\Psi) = |\langle \phi, \Psi \rangle|^2$$

emerges as the unique transition geometry compatible with:

1. SU(2)-covariant closure rotations,
2. global-phase invariance,
3. continuity,
4. orthogonality,
5. and projective distinguishability on the scalar-time closure manifold.

Thus the quadratic Born-type structure appearing in the Bell derivation is recovered as a geometric consequence of admissible scalar-time closure geometry.

J Appendix J: Derivation of the Antisymmetric Closure State from Pairwise Scalar-Time Closure

The purpose of this appendix is to derive the antisymmetric correlated closure state used in the Bell analysis from closure principles rather than introducing it axiomatically.

In the main text the correlated state

$$\Psi_S = \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_- - \psi_- \otimes \psi_+)$$

was employed in the Bell derivation. However, if such a state is merely inserted by hand, the resulting Bell geometry risks appearing postulated rather than emergent.

The objective here is therefore to show that antisymmetric pairwise closure arises naturally from:

1. indistinguishable closure pairing,
2. rotational invariance,
3. and total closure-neutrality constraints.

The derivation does not assume fermionic statistics as an independent axiom. Rather, antisymmetry emerges as the unique globally closure-balanced two-sector state.

J.1 Two Closure Sectors

Let

$$\mathcal{H}_A = \mathrm{span}\{\psi_+, \psi_-\},$$

$$\mathcal{H}_B = \mathrm{span}\{\psi_+, \psi_-\},$$

be identical degenerate scalar-time closure sectors.

The composite admissible sector is:

$$\mathcal{H}_A \otimes \mathcal{H}_B.$$

A general normalized two-sector state may be written:

$$\Psi = \alpha\, \psi_+ \otimes \psi_+ + \beta\, \psi_+ \otimes \psi_- + \gamma\, \psi_- \otimes \psi_+ + \delta\, \psi_- \otimes \psi_-.$$

Normalization requires:

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1.$$

J.2 Closure Neutrality

The Bell geometry considered in the main text concerns maximally anti-correlated closure pairing.

We therefore impose total closure neutrality.

Define the closure-orientation operator:

$$\Sigma_z = \sigma_z.$$

The total closure orientation is:

$$\Sigma_{\mathrm{tot}} = \Sigma_z \otimes I + I \otimes \Sigma_z.$$

Its action on basis states is:

$$\Sigma_{\mathrm{tot}}(\psi_+ \otimes \psi_+) = 2(\psi_+ \otimes \psi_+),$$

$$\Sigma_{\mathrm{tot}}(\psi_- \otimes \psi_-) = -2(\psi_- \otimes \psi_-),$$

$$\Sigma_{\mathrm{tot}}(\psi_+ \otimes \psi_-) = 0,$$

$$\Sigma_{\mathrm{tot}}(\psi_- \otimes \psi_+) = 0.$$

Thus total closure-neutral states belong to:

$$\mathrm{span}\{\psi_+ \otimes \psi_-, \psi_- \otimes \psi_+\}.$$

Consequently:

$$\alpha = \delta = 0.$$

The admissible closure-neutral state therefore reduces to:

$$\Psi = \beta\,\psi_+ \otimes \psi_- + \gamma\,\psi_- \otimes \psi_+.$$

J.3 Exchange Symmetry

The two closure sectors are physically indistinguishable.

Define the exchange operator:

$$P_{12}(\phi \otimes \chi) = \chi \otimes \phi.$$

Then:

$$P_{12}(\psi_+ \otimes \psi_-) = \psi_- \otimes \psi_+,$$

$$P_{12}(\psi_- \otimes \psi_+) = \psi_+ \otimes \psi_-.$$

Thus:

$$P_{12}\Psi = \beta\,\psi_- \otimes \psi_+ + \gamma\,\psi_+ \otimes \psi_-.$$

The eigenstates of exchange are therefore:

Symmetric:

$$\Psi_+ = \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_- + \psi_- \otimes \psi_+),$$

Antisymmetric:

$$\Psi_- = \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_- - \psi_- \otimes \psi_+).$$

J.4 Rotational Closure Invariance

We now determine which combination remains globally closure-invariant.

Define the total closure rotation generators:

$$J_i = \frac{1}{2}(\sigma_i \otimes I + I \otimes \sigma_i).$$

A rotationally invariant closure state must satisfy:

$$J_i\Psi = 0, \qquad i = x, y, z.$$

This is the closure analogue of vanishing total angular momentum.

J.5 Action on Symmetric and Antisymmetric States

Consider first:

$$\Psi_+ = \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_- + \psi_- \otimes \psi_+).$$

Acting with:

$$J_x = \frac{1}{2}(\sigma_x \otimes I + I \otimes \sigma_x),$$

gives:

$$J_x\Psi_+ = \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_+ + \psi_- \otimes \psi_-).$$

Hence:

$$J_x\Psi_+ \neq 0.$$

Similarly:

$$J_y\Psi_+ \neq 0, \qquad J_z\Psi_+ = 0.$$

Therefore the symmetric state is not rotationally closure-invariant.

Now consider:

$$\Psi_- = \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_- - \psi_- \otimes \psi_+).$$

Direct computation gives:

$$J_x\Psi_- = 0,$$

$$J_y\Psi_- = 0,$$

$$J_z\Psi_- = 0.$$

Thus:

$$\boldsymbol{J}\Psi_- = 0.$$

Therefore the antisymmetric state is the unique rotationally invariant closure-neutral two-sector state.

J.6 Uniqueness

We now prove uniqueness.

Proposition 4. *The antisymmetric closure state*

$$\Psi_- = \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_- - \psi_- \otimes \psi_+)$$

is the unique normalized two-sector state satisfying:

1. *closure neutrality,*
2. *exchange admissibility,*
3. *and global rotational closure invariance.*

Proof. Closure neutrality restricts admissible states to:

$$\mathrm{span}\{\psi_+ \otimes \psi_-, \psi_- \otimes \psi_+\}.$$

Exchange decomposition splits this subspace into symmetric and antisymmetric sectors.

Direct action of the total closure generators shows:

$$\boldsymbol{J}\Psi_+ =\neq 0,$$

while:

$$\boldsymbol{J}\Psi_- = 0.$$

Hence only the antisymmetric combination remains rotationally invariant.

Since the neutral subspace is two-dimensional and decomposes uniquely into symmetric and antisymmetric sectors, the antisymmetric state is unique up to global phase.

□

J.7 Emergence of the Bell State

The antisymmetric closure state therefore emerges not from independent insertion, but from:

$$\text{closure neutrality} + \text{exchange indistinguishability} + \text{rotational closure invariance}.$$

This is precisely the state used in the Bell derivation:

$$\Psi_S = \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_- - \psi_- \otimes \psi_+).$$

Thus the Bell geometry of the main text is not produced by assuming the singlet state a priori. Instead, the singlet-like closure state appears as the unique globally closure-balanced correlated sector.

J.8 Physical Interpretation in TSFT

Within the scalar-time framework, the antisymmetric closure state represents a pairwise coherence configuration whose total closure orientation vanishes globally while preserving maximal relative correlation internally.

The resulting structure possesses:

1. zero net closure orientation,
2. exact rotational invariance,
3. and maximal projective anti-correlation.

Consequently, Bell-type correlations emerge not from hidden-variable assignment, but from the geometry of globally balanced closure sectors on the admissible scalar-time manifold.

J.9 Relation to Standard Quantum Structure

The derivation presented here parallels the emergence of the spin-singlet sector in ordinary quantum mechanics, but differs conceptually in its starting point.

Standard quantum theory introduces:

1. tensor-product Hilbert spaces,
2. spin representations,
3. and singlet states

axiomatically.

By contrast, the TSFT derivation begins from:

1. degenerate scalar-time closure sectors,
2. admissible closure neutrality,
3. exchange indistinguishability,
4. and rotational closure preservation.

The antisymmetric Bell state then emerges as the unique admissible globally invariant closure configuration.

This appendix therefore closes the final major structural gap in the Bell derivation developed throughout the paper.

K Appendix K: Emergence of Complex Closure Amplitudes from Scalar-Time Recurrence

The purpose of this appendix is to address an important structural question left implicit in the main text:

> Why do admissible scalar-time closure sectors possess complex amplitude structure rather than purely real state geometry?

This issue is fundamental because:

1. phase recurrence,
2. interference structure,
3. unitary evolution,
4. SU(2) geometry,
5. and Bell correlations

all depend intrinsically on complex projective structure.

If complex amplitudes are merely inserted axiomatically, then the closure program remains incomplete.

The goal of the present appendix is therefore to show that complex closure amplitudes arise naturally from:

1. self-adjoint scalar-time generators,
2. recurrent oscillatory closure evolution,
3. and continuous closure-preserving dynamics.

We do not claim here to derive the entirety of complex analysis from first principles. Rather, we show that continuous recurrent scalar-time evolution naturally induces a complex unitary representation.

K.1 Real Scalar-Time Closure Dynamics

The emergence of complex closure amplitudes may be understood starting from a purely real scalar-time fluctuation field.

Let

$$\psi(x, \tau) \in \mathbb{R}$$

be a real-valued scalar-time fluctuation defined on a spatial Cauchy slice

$$\Sigma_t,$$

with scalar-time evolution parameter

$$\tau.$$

After separation of scalar-time evolution from spatial mode structure, admissible closure modes satisfy the spectral equation

$$L_\Theta u = \lambda u,$$

where

$$L_\Theta = -\nabla^2 + V''(\Theta_0)$$

acts on

$$L^2(\Sigma_t, d^3x).$$

The associated scalar-time evolution equation takes the real second-order form

$$\frac{d^2u}{d\tau^2} + L_\Theta u = 0.$$

This is the infinite-dimensional analogue of a harmonic oscillator equation.

For a single closure eigenmode

$$L_\Theta u_n = \lambda_n u_n,$$

the scalar-time evolution reduces to

$$\frac{d^2q_n}{d\tau^2} + \lambda_n q_n = 0,$$

whose general real solution is

$$q_n(\tau) = A_n \cos(\omega_n \tau) + B_n \sin(\omega_n \tau),$$

with

$$\omega_n = \sqrt{\lambda_n}.$$

Thus every admissible closure mode possesses an intrinsic recurrent phase structure even before the introduction of complex amplitudes.

The phase-space evolution of the pair

$$(q_n, p_n) = \left(q_n, \frac{dq_n}{d\tau}\right)$$

is generated by the symplectic rotation matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

satisfying

$$J^2 = -I.$$

Therefore the real closure dynamics naturally possess an intrinsic rotational structure equivalent to multiplication by the imaginary unit:

$$J \longleftrightarrow i.$$

This does not yet uniquely determine a complex representation. Rather, it establishes the existence of a compatible complex structure associated with recurrent scalar-time phase rotation.

Once closure norm preservation is imposed through the

$$L^2(\Sigma_t)$$

inner product,

$$\langle u, v \rangle = \int_{\Sigma_t} d^3x\, u^*(x) v(x),$$

the recurrent closure dynamics acquire a compatible metric and symplectic structure. The resulting phase geometry then admits the standard complex representation

$$u_n(\tau) \sim e^{-i\omega_n \tau}.$$

Thus complex scalar-time amplitudes are not inserted as independent quantum axioms. They arise as the minimal algebraic representation of recurrent closure-preserving phase evolution generated by the real scalar-time dynamics themselves.

K.2 Oscillatory Closure Evolution

Consider a single eigenmode:

$$L_\Theta \psi_n = \lambda_n \psi_n, \qquad \lambda_n > 0.$$

Then:

$$\partial_\tau^2 \psi_n = -\lambda_n \psi_n.$$

Solutions are:

$$\psi_n(\tau) = A_n \cos(\omega_n \tau) + B_n \sin(\omega_n \tau),$$

with:

$$\omega_n = \sqrt{\lambda_n}.$$

Thus admissible closure evolution is intrinsically oscillatory.

The pair:

$$(\cos \omega\tau, \sin \omega\tau)$$

forms a rotational phase plane.

K.3 Phase-Space Rotation Structure

Define the closure phase vector:

$$\Phi_n(\tau) = \begin{pmatrix} q_n(\tau) \\ p_n(\tau) \end{pmatrix},$$

where:

$$q_n(\tau) = \psi_n(\tau),$$

$$p_n(\tau) = \frac{1}{\omega_n} \partial_\tau \psi_n(\tau).$$

The evolution equations become:

$$\partial_\tau q_n = \omega_n p_n,$$

$$\partial_\tau p_n = -\omega_n q_n.$$

Hence:

$$\partial_\tau \begin{pmatrix} q_n \\ p_n \end{pmatrix} = \omega_n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q_n \\ p_n \end{pmatrix}.$$

Define:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since:

$$J^2 = -I,$$

the phase-plane generator possesses the algebraic structure of multiplication by i.

Thus recurrent closure evolution naturally induces a complex structure on the phase plane.

K.4 Complex Closure Representation

Define the complex closure amplitude:

$$z_n(\tau) = q_n(\tau) + ip_n(\tau).$$

Then:

$$\partial_\tau z_n = -i\omega_n z_n.$$

Hence:

$$z_n(\tau) = e^{-i\omega_n \tau} z_n(0).$$

The oscillatory closure evolution therefore becomes a complex phase rotation.

Importantly, this complex structure has not been postulated independently. It emerges from:

1. recurrent scalar-time oscillation,
2. phase-plane rotation,
3. and the algebraic property:

$$J^2 = -I.$$

K.5 Stone-Type Unitary Evolution

We now extend this structure to the full closure sector.

Since L_Θ is self-adjoint, Stone's theorem implies:

$$U(\tau) = e^{-iL_\Theta \tau}$$

is a strongly continuous unitary group.

The appearance of i is not arbitrary.

A continuous norm-preserving one-parameter evolution group on a Hilbert-type closure space necessarily possesses an anti-self-adjoint generator:

$$\partial_\tau U(\tau) = -iL_\Theta U(\tau).$$

The complex phase factor therefore arises naturally from continuous closure-preserving recurrence dynamics.

K.6 Why Real Closure Geometry Is Insufficient

Suppose closure amplitudes were restricted to purely real values.

Then admissible transformations preserving normalization would reduce to:

$$O(2)$$

rather than:

$$U(2).$$

However:

1. real orthogonal geometry cannot support continuous phase interference,
2. cannot produce complex projective structure,
3. and cannot generate the full Bloch-sphere phase geometry required for Bell correlations.

In particular, interference terms:

$$|A+B|^2 = |A|^2 + |B|^2 + 2\mathrm{Re}(A^*B)$$

require relative complex phase structure.

Thus the experimentally observed closure-interference geometry forces complex amplitude representation.

K.7 Complex Projective Closure Geometry

Once recurrent closure evolution induces complex amplitudes:

$$\Psi = c_1\psi_1 + c_2\psi_2, \qquad c_i \in \mathbb{C},$$

the normalized closure sector becomes:

$$S^3 \subset \mathbb{C}^2.$$

Global phase redundancy:

$$\Psi \sim e^{i\alpha}\Psi$$

then produces:

$$S^3/U(1) = \mathbb{CP}^1.$$

Thus the full projective closure geometry used in the Bell derivation follows naturally from:

$$\text{oscillatory recurrence} + \text{self-adjoint generators} + \text{continuous norm-preserving evolution.}$$

K.8 Closure-Phase Interpretation

Within the TSFT framework, complex phase therefore acquires a direct physical interpretation.

The quantity:

$$e^{-i\omega\tau}$$

represents scalar-time closure rotation within the recurrent phase plane of admissible coherence evolution.

Complex amplitudes are therefore not abstract bookkeeping devices. They encode:

1. closure orientation,

2. recurrence phase,

3. and interference structure

within the scalar-time manifold.

K.9 Main Result

We may summarize the appendix as follows.

Theorem 1 (Emergence of Complex Closure Amplitudes). *Let admissible scalar-time closure sectors possess:*

1. self-adjoint recurrence generators,

2. continuous norm-preserving evolution,

3. and oscillatory recurrent closure structure.

Then the resulting phase-plane dynamics naturally induce a complex unitary representation of closure evolution.

K.10 Consequences for the Bell Derivation

This appendix closes an important structural gap in the Bell program developed throughout the paper.

Specifically:

1. recurrent scalar-time closure induces complex phase geometry;

2. complex closure amplitudes produce projective state space;

3. projective closure geometry yields SU(2)-covariant rotational structure;

4. and the resulting projector geometry produces Bell correlations and the Tsirelson bound.

Thus the complex geometry underlying the Bell derivation is not inserted independently, but emerges from recurrent scalar-time closure evolution itself.

L Appendix L: Tensor Product Composition of Independent Scalar-Time Closure Sectors

The purpose of this appendix is to justify the use of the composite sector

$$\mathcal{H}_A \otimes \mathcal{H}_B$$

in the Bell derivation.

The tensor product should not be treated as an arbitrary import from standard quantum mechanics. In the present framework, it arises as the natural composition law for independently admissible scalar-time closure sectors when the joint system must preserve bilinearity, independent normalization, local closure operations, and product-state separability.

L.1 Independent Closure Sectors

Let

$$\mathcal{H}_A$$

and

$$\mathcal{H}_B$$

be two independently admissible scalar-time closure sectors.

Each sector consists of recurrent closure modes satisfying:

$$L_A\psi_i = \lambda_i\psi_i,$$

$$L_B\chi_j = \mu_j\chi_j.$$

Let

$$\{\psi_i\}_{i=1}^n$$

be an orthonormal closure basis for $\mathcal{H}_A$, and let

$$\{\chi_j\}_{j=1}^m$$

be an orthonormal closure basis for $\mathcal{H}_B$.

A composite closure configuration must encode all pairwise joint possibilities:

$$(\psi_i, \chi_j).$$

Thus the joint basis must contain nm independent elementary closure combinations.

L.2 Bilinear Composition Requirement

The composition map

$$C : \mathcal{H}_A \times \mathcal{H}_B \to \mathcal{H}_{AB}$$

must satisfy bilinearity.

That is, for closure amplitudes $a, b \in \mathbb{C}$,

$$C(a\psi_1 + b\psi_2, \chi) = aC(\psi_1, \chi) + bC(\psi_2, \chi),$$

and

$$C(\psi, a\chi_1 + b\chi_2) = aC(\psi, \chi_1) + bC(\psi, \chi_2).$$

Bilinearity is required because closure amplitudes are linear spectral amplitudes, and independent superpositions in one sector must distribute over independent closure states in the other sector.

This is the universal defining property of the tensor product.

Therefore there exists a vector space

$$\mathcal{H}_A \otimes \mathcal{H}_B$$

and a bilinear map

$$\otimes : \mathcal{H}_A \times \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$$

such that every bilinear composition factors uniquely through it.

Thus:

$$C(\psi, \chi) = \psi \otimes \chi.$$

L.3 Product Closure States

For independent states

$$\Psi_A = \sum_i a_i \psi_i,$$

$$\Psi_B = \sum_j b_j \chi_j,$$

the composite product closure state is:

$$\Psi_{AB} = \Psi_A \otimes \Psi_B.$$

By bilinearity:

$$\Psi_{AB} = \sum_{i,j} a_i b_j \, \psi_i \otimes \chi_j.$$

Thus joint closure amplitudes factor:

$$c_{ij} = a_i b_j.$$

This is the mathematical expression of independent scalar-time closure composition.

L.4 Inner Product on the Composite Sector

The composite inner product must preserve independent closure amplitudes.

For product states, require:

$$\langle \psi_i \otimes \chi_j, \psi_k \otimes \chi_\ell \rangle = \langle \psi_i, \psi_k \rangle_A \, \langle \chi_j, \chi_\ell \rangle_B.$$

Using orthonormality:

$$\langle \psi_i, \psi_k \rangle_A = \delta_{ik},$$

$$\langle \chi_j, \chi_\ell \rangle_B = \delta_{j\ell},$$

we obtain:

$$\langle \psi_i \otimes \chi_j, \psi_k \otimes \chi_\ell \rangle = \delta_{ik}\delta_{j\ell}.$$

Thus the product basis is orthonormal.

For general states:

$$\Phi_{AB} = \sum_{i,j} c_{ij} \psi_i \otimes \chi_j,$$

the norm is:

$$\|\Phi_{AB}\|^2 = \sum_{i,j} |c_{ij}|^2.$$

This preserves closure normalization in the composite sector.

L.5 Independent Evolution

Let scalar-time evolution on each sector be:

$$U_A(\tau) = e^{-iL_A\tau},$$

$$U_B(\tau) = e^{-iL_B\tau}.$$

For independent sectors, joint evolution must satisfy:

$$U_{AB}(\tau)(\psi \otimes \chi) = U_A(\tau)\psi \otimes U_B(\tau)\chi.$$

Therefore:

$$U_{AB}(\tau) = U_A(\tau) \otimes U_B(\tau).$$

If:

$$L_A\psi_i = \lambda_i\psi_i,$$

$$L_B\chi_j = \mu_j\chi_j,$$

then:

$$U_{AB}(\tau)(\psi_i \otimes \chi_j) = e^{-i(\lambda_i+\mu_j)\tau}\psi_i \otimes \chi_j.$$

Thus the composite generator is:

$$L_{AB} = L_A \otimes I + I \otimes L_B.$$

This is the unique additive closure generator compatible with independent scalar-time recurrence.

L.6 Local Closure Operations

A local closure operation on sector A must act as:

$$M_A \otimes I_B.$$

A local closure operation on sector B must act as:

$$I_A \otimes M_B.$$

These operations commute:

$$[M_A \otimes I_B,\, I_A \otimes M_B] = 0.$$

This expresses the independence of local closure transformations before correlation constraints are imposed.

Thus tensor composition naturally preserves:

1. independent sector identity,
2. local operation structure,
3. joint normalization,
4. and additive scalar-time evolution.

L.7 Emergence of Correlated Closure States

The tensor product sector contains product states:

$$\Psi_A \otimes \Psi_B,$$

but it also contains non-factorizable states:

$$\Phi_{AB} = \sum_{i,j} c_{ij} \psi_i \otimes \chi_j$$

for which no decomposition

$$c_{ij} = a_i b_j$$

exists.

Such states represent correlated closure configurations.

In the Bell analysis, the relevant correlated state is:

$$\Psi_S = \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_- - \psi_- \otimes \psi_+).$$

This state is non-factorizable, but it belongs naturally to the composite closure sector because the tensor product is the admissible bilinear composition space of the two independent closure sectors.

L.8 Closure Composition Theorem

Theorem 2 (Tensor Product Closure Composition)**.** *Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be independently admissible scalar-time closure sectors. Suppose their composite sector must satisfy:*

1. *bilinear composition of closure amplitudes,*
2. *preservation of independent normalization,*
3. *local closure operations acting independently,*
4. *and additive scalar-time evolution for uncoupled sectors.*

Then the admissible composite closure space is:

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B.$$

Proof. Bilinearity of closure amplitudes requires a universal bilinear composition space. By the universal property of the tensor product, this space is $\mathcal{H}_A \otimes \mathcal{H}_B$.

Preservation of independent normalization fixes the product inner product:

$$\langle \psi \otimes \chi, \phi \otimes \eta \rangle = \langle \psi, \phi \rangle_A \langle \chi, \eta \rangle_B.$$

Independent local operations require:

$$M_A \mapsto M_A \otimes I_B, \qquad M_B \mapsto I_A \otimes M_B.$$

Independent scalar-time evolution requires:

$$U_{AB}(\tau) = U_A(\tau) \otimes U_B(\tau),$$

with generator:

$$L_{AB} = L_A \otimes I + I \otimes L_B.$$

These conditions uniquely characterize tensor product composition.

□

L.9 Physical Interpretation in TSFT

Within TSFT, the tensor product does not enter as an arbitrary quantum postulate.

It expresses the composition of independently recurrent informational closure sectors carried by the scalar-time field.

Product states represent independently persisting closure structures. Non-factorizable states represent globally constrained closure configurations whose admissibility cannot be reduced to either sector alone.

Thus entanglement is interpreted as non-factorizable scalar-time closure correlation.

The Bell state used in the main text is therefore not imposed on an externally assumed quantum tensor product. Rather, it belongs to the natural bilinear composition space required by independent scalar-time closure sectors.

L.10 Consequence for Bell Geometry

The Bell derivation requires:

$$\mathcal{H}_A \otimes \mathcal{H}_B,$$

not because standard quantum mechanics assumes it, but because:

$$\text{independent closure sector} + \text{independent closure sector}$$

requires a bilinear composite amplitude space.

Once this space exists, closure neutrality and rotational invariance select the antisymmetric correlated state derived in Appendix J.

Therefore the chain becomes:

$$\Theta \to \text{closure sectors} \to \text{tensor composition} \to \text{antisymmetric closure state} \to E(a,b) = -a\cdot b \to S_{\max} = 2\sqrt{2}.$$

This closes the tensor-product gap in the Bell derivation.

M Appendix M: Compressed Derivation of Complex Scalar-Time Phase Structure from Prior TSFT Results

M.1 Purpose of the Appendix

The derivation in the main text uses complex closure amplitudes and unitary phase evolution. This appendix explains why that structure is not imported from quantum mechanics as an independent postulate. Rather, it is the compressed consequence of the scalar-time spectral program developed in prior TSFT work: localized scalar-time fluctuation modes, recurrent phase closure, temporal-efficiency partition, bound-state spectral discreteness, and coherence locking.

The purpose is not to rederive the full atomic, nuclear, or scalar-potential program. The purpose is narrower: to show why admissible scalar-time states naturally acquire a complex phase representation before the Bell-sector geometry is considered.

M.2 Scalar-Time Fluctuations Produce Oscillatory Eigenmodes

TSFT begins from the scalar-time field

$$\Theta = \Theta(x,t),$$

governed by an action of the form

$$S[\Theta] = \int d^4x \left[\frac{1}{2}\partial_\mu\Theta\,\partial^\mu\Theta - V(\Theta)\right].$$

Let

$$\Theta_0(x)$$

be a stable background solution. Small fluctuations are written as

$$\Theta(x,t) = \Theta_0(x) + \psi(x,t).$$

Linearization gives a fluctuation equation of the form

$$-\nabla^2\psi + V''(\Theta_0)\psi = \omega^2\psi.$$

For stationary coherence modes one separates temporal and spatial dependence:

$$\psi(x,t) = \psi(x)e^{-i\omega t}.$$

This is the first appearance of complex phase structure. It is not a quantum-mechanical postulate. It is the minimal representation of recurrent scalar-time oscillation.

M.3 Why the Complex Representation Is Forced by Recurrence

A real oscillatory scalar-time mode may be written as

$$\psi(x,t) = \psi(x)\cos(\omega t + \phi).$$

However, phase evolution is compositionally simpler and structurally complete when represented as

$$e^{-i\omega t}.$$

Indeed,

$$e^{-i\omega(t_1+t_2)} = e^{-i\omega t_1}e^{-i\omega t_2}.$$

Thus scalar-time recurrence requires a one-parameter phase group:

$$U(1) = \{e^{i\alpha} : \alpha \in \mathbb{R}\}.$$

The complex number is therefore not an added ontology. It is the irreducible algebraic representation of phase-return composition.

Equivalently, every complex phase may be represented by a real two-dimensional rotation:

$$e^{-i\omega t} \quad \longleftrightarrow \quad \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}.$$

Thus complex scalar-time amplitudes are the compact representation of real recurrent two-component phase rotation.

M.4 Temporal-Efficiency Partition Supplies the Physical Meaning of Phase

Prior TSFT work introduced the temporal-efficiency partition

$$\eta_{\mathrm{int}}^2 + \eta_{\mathrm{prop}}^2 = 1,$$

where η_{int} represents internally retained temporal evolution and η_{prop} represents propagative temporal allocation. The null sector satisfies

$$\eta_{\mathrm{int}} = 0, \qquad \eta_{\mathrm{prop}} = 1.$$

Massive and bound structures require

$$\eta_{\mathrm{int}} > 0.$$

Therefore internal scalar-time phase is not a decorative mathematical label. It represents retained temporal evolution.

Prior scalar-potential work used this partition to constrain the global structure of $V(\Theta)$, including the null-sector conditions and the closed potential

$$V(\Theta) = V_* \left(e^{2\Theta} - 1 - 2\Theta - 2\Theta^2\right).$$

The physical interpretation is therefore:

$$\text{internal temporal retention} \quad \Longrightarrow \quad \text{phase evolution} \quad \Longrightarrow \quad \text{complex amplitude representation.}$$

M.5 Bound-State Spectra Require Normalizable Phase Modes

In the atomic spectral sector, localized scalar-time backgrounds satisfy asymptotically

$$\Theta_0(r) = \Theta_\infty + \frac{A}{r}.$$

Expanding the fluctuation operator about this background yields an effective radial operator whose leading inverse-radial term produces a normalizable bound-state spectrum:

$$\epsilon_n = -\frac{\kappa^2}{4n^2}.$$

The admissible states are therefore not arbitrary functions. They are normalizable eigenmodes selected by scalar-time closure conditions.

The corresponding state form is

$$\psi_{n\ell m}(x,t) = R_{n\ell}(r) Y_{\ell m}(\theta,\phi) e^{-i\omega_n t}.$$

Thus the full scalar-time mode already contains:

$$\text{spatial eigenfunction } \times \text{ complex temporal phase.}$$

This same structure underlies the later Bell-sector construction. The Bell paper does not introduce complex amplitudes anew; it restricts attention to a degenerate two-dimensional closure subspace of an already phase-bearing scalar-time spectrum.

M.6 Subshell and Nuclear Results Confirm That Phase Coherence Is Structural

Prior TSFT atomic-periodicity work showed that subleading scalar-time curvature corrections produce angular-momentum-dependent spectral shifts of the form

$$\epsilon_{n\ell} = -\frac{\kappa^2}{4n^2} - \frac{\beta\kappa^2}{4n^3(\ell + \frac{1}{2})}.$$

There, the coefficient β was connected to temporal allocation through

$$\beta = B\eta_{\text{config}}^2.$$

Thus spectral organization depends on internal temporal coherence, not merely on spatial boundary conditions.

Likewise, the nuclear-structure papers treat particles and nucleons as coherence-stable scalar-time eigenmodes and composites. In that framework, mass emerges from intrinsic scalar-time frequency:

$$mc^2 = \hbar\omega_0,$$

and rest energy follows as

$$E = mc^2.$$

Therefore the same chain appears repeatedly:

$$\text{scalar-time field} \to \text{oscillatory eigenmode} \to \text{phase recurrence} \to \text{mass/coherence structure} \to \text{spectral admissibilit}$$

M.7 Degenerate Closure Sectors Inherit Complex Amplitudes

Let two admissible scalar-time modes share the same closure eigenvalue:

$$L_\Theta\psi_1 = \lambda\psi_1, \qquad L_\Theta\psi_2 = \lambda\psi_2.$$

Because both modes possess recurrent scalar-time phase, a general admissible state in the degenerate sector is

$$\Psi = c_1\psi_1 + c_2\psi_2,$$

where

$$c_1, c_2 \in \mathbb{C}.$$

The complex coefficients encode relative phase and amplitude between recurrent closure modes. They are required because the physically relevant information is not merely which mode is present, but how the modes are phase-aligned under scalar-time recurrence.

Scalar-time evolution gives

$$U(\tau)\Psi = e^{-i\lambda\tau}\Psi.$$

Thus the entire degenerate sector evolves by a common phase, while the relative complex orientation

$$(c_1, c_2)$$

remains physically meaningful.

M.8 Why Unitary Transformations Follow

Closure-preserving transformations must preserve total closure amplitude:

$$\langle \Psi, \Psi \rangle.$$

For a transformation

$$\Psi \mapsto U\Psi,$$

closure preservation requires

$$\langle U\Psi, U\Phi \rangle = \langle \Psi, \Phi \rangle$$

for arbitrary admissible states Ψ, Φ. Therefore

$$U^\dagger U = I.$$

Hence the admissible internal transformation group is unitary.

For a two-dimensional degenerate closure sector,

$$U \in U(2).$$

Removing physically irrelevant global phase leaves the spinorial closure geometry governed by

$$SU(2),$$

with physical rays forming

$$\mathbb{CP}^1 \cong S^2.$$

Thus the complex two-state geometry used in the Bell derivation follows from the prior scalar-time spectral chain.

M.9 Compressed Chain of Dependence

The logical dependence may be summarized as follows:

$$\Theta(x,t) \quad \Longrightarrow \quad L_\Theta \psi = \lambda \psi$$

$$L_\Theta \psi = \lambda \psi \quad \Longrightarrow \quad \psi(x,\tau) = \psi(x) e^{-i\lambda\tau}$$

$$e^{-i\lambda\tau} \quad \Longrightarrow \quad U(1) \text{ phase recurrence}$$

$$U(1) \text{ phase recurrence} \quad \Longrightarrow \quad \mathbb{C}\text{-valued closure amplitudes}$$

$$\mathbb{C}\text{-valued closure amplitudes} + \text{norm preservation} \quad \Longrightarrow \quad U(n) \text{ closure transformations}$$

$$n = 2 \quad \Longrightarrow \quad SU(2) \text{ spinorial closure geometry}$$

$$SU(2) + \text{global phase quotient} \quad \Longrightarrow \quad \mathbb{CP}^1 \cong S^2.$$

This is precisely the two-state geometry required for Bell correlations.

M.10 Conclusion

The complex structure used in the Bell-sector derivation is not an independent quantum axiom. It is inherited from scalar-time recurrence.

Prior TSFT work already established that admissible physical structures arise as normalizable, coherence-stable, oscillatory eigenmodes of scalar-time dynamics. Such modes necessarily carry phase. Phase recurrence composes according to $U(1)$. The complex representation is the minimal algebraic representation of that recurrence. Degenerate closure sectors then inherit complex amplitude geometry, and closure-preserving transformations become unitary.

Therefore the Bell paper does not assume Hilbert-space quantum mechanics at the start. It begins from a scalar-time spectral framework in which complex phase geometry has already emerged from recurrent temporal coherence.