Atomic Spectral Structure from Time-Scalar Field Theory

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Abstract

We derive the structural form of atomic bound-state spectra from Time-Scalar Field Theory (TSFT) without postulating quantum mechanics or a Coulomb potential. Starting from the scalar-time field equation, we construct localized, static background solutions with asymptotic behavior $\Theta_0(r) = \Theta_\infty + A/r$. Expanding the induced operator $V''(\Theta_0(r))$ about Θ_∞ yields an emergent inverse-radial interaction $-\kappa/r$ at large distances, where κ is fixed by the background amplitude and the third derivative of the scalar-time potential.

The absolute normalization of κ is obtained from the recovered electrodynamic sector of TSFT. Using $E_\Theta = -\nabla\Theta$ together with the inhomogeneous Maxwell equation $\nabla \cdot E_\Theta = \rho_q/\varepsilon_0$, Gauss-law normalization determines the far-field coefficient A in terms of the conserved source charge. This yields $\kappa = CZ\alpha$, with C fixed by the scalar-time potential curvature and electrodynamic normalization, and α inherited from the closure structure of the theory.

The resulting radial problem is a Sturm–Liouville system with inverse-radial leading behavior. Imposing normalizability yields Laguerre termination and a discrete spectrum $\varepsilon_n = -\kappa^2/(4n^2)$ with shell capacities $2n^2$. Subleading $1/r^2$ corrections generate subshell-dependent shifts $\Delta_{n\ell}$ treated perturbatively.

Thus, the principal organization of atomic spectra arises from scalar-time field dynamics together with the internally recovered electrodynamic normalization, providing a non-circular route to the hydrogenic spectral structure.

1 Introduction

Atomic structure exhibits a hierarchy of discrete bound states organized into shells. These shells determine the chemical behavior of elements and give rise to the periodic organization observed in the periodic table. In conventional theory, this structure is obtained by solving quantum wave equations in a central potential, with shell capacities arising from angular momentum degeneracy and fermionic filling.

While this framework is successful, it relies on several foundational inputs that are not derived within the theory itself. These include the form of the central potential, the structure of the governing wave equation, and the definition of the state space in which solutions reside. As a result, atomic spectral structure is obtained within a formalism rather than derived from a more primitive underlying field description.

Time-Scalar Field Theory (TSFT) provides such a primitive description. In TSFT, time is promoted to a scalar field

$$\Theta(x, t),$$

and physical structure arises from the spatial and temporal variation of this field. Stable excitations correspond to coherence-preserving eigenmodes of the scalar-time operator, and composite structures emerge through interactions between these modes.

Previous work established that fermionic excitations arise from the spectral structure of the scalar-time field, and that multi-fermion coherence locking produces composite states corresponding to nucleons. Interactions between these composites give rise to nuclear structure, including shell closures determined by spin-curvature coupling and spectral ordering.

The next structural level is atomic organization. The central question addressed in this work is:

Can the bound-state spectrum of atomic systems be derived directly from scalar-time field dynamics

To answer this question, we construct a scalar-time background corresponding to a spatially localized source and analyze the excitation spectrum of perturbations about this background. The approach follows a strictly derivational path:

1. A static, spatially localized scalar-time background is defined from the field equation.
2. The scalar-time field is linearized about this background to obtain a fluctuation operator.
3. The admissible excitation modes are defined as eigenfunctions of this operator.
4. Rotational symmetry of the background is used to derive angular momentum structure and degeneracy.
5. The resulting radial equation determines the bound-state spectrum.

No external potentials or quantum-mechanical postulates are introduced. All structure arises from the scalar-time field equation and its symmetries.

This procedure yields a hierarchy of discrete states organized by angular momentum and radial structure. The capacities of these states follow from degeneracy and fermionic occupation, forming the basis for atomic shell structure.

The goal of this work is therefore not to reproduce known results within an assumed framework, but to derive the existence and organization of atomic bound states directly from scalar-time field dynamics.

2 Scalar-Time Field and Background Configuration

We begin from the scalar-time field action

$$S_\Theta = \int d^4x \left[\frac{1}{2} \partial_\mu \Theta \, \partial^\mu \Theta - V(\Theta) \right], \tag{1}$$

where $\Theta(x,t)$ is the scalar-time field and $V(\Theta)$ is a scalar potential governing its local dynamics.

Variation of the action with respect to Θ yields the field equation

$$\Box \Theta = V'(\Theta), \tag{2}$$

where $\Box = \partial_\mu \partial^\mu$ is the d'Alembert operator.

We consider configurations corresponding to stationary bound systems. In such systems, the scalar-time field admits a static background solution $\Theta_0(\mathbf{x})$ satisfying

$$\nabla^2 \Theta_0(\mathbf{x}) = V'\big(\Theta_0(\mathbf{x})\big), \tag{3}$$

where time derivatives vanish.

We restrict attention to spatially localized configurations generated by a compact source. These configurations are characterized by:

1. Finite total energy
2. Spatial localization
3. Asymptotic approach to a constant background value

Under these conditions, the scalar-time field satisfies

$$\Theta_0(\mathbf{x}) \to \Theta_\infty \quad \text{as} \quad |\mathbf{x}| \to \infty, \tag{4}$$

where Θ_∞ is a constant.

In the absence of preferred spatial directions, the lowest-energy configuration is rotationally symmetric. Therefore, the background field depends only on the radial coordinate:

$$\Theta_0(\mathbf{x}) = \Theta_0(r), \quad r = |\mathbf{x}|. \tag{5}$$

Substitution into the field equation yields the radial equation

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Theta_0}{dr} \right) = V'\big(\Theta_0(r)\big). \tag{6}$$

To determine the behavior of the field at large distances, we consider the asymptotic regime in which the field approaches its background value. Writing

$$\Theta_0(r) = \Theta_\infty + \delta\Theta(r), \tag{7}$$

with $|\delta\Theta| \ll 1$, we expand the potential:

$$V'\big(\Theta_0\big) = V'(\Theta_\infty) + V''(\Theta_\infty)\delta\Theta + \mathcal{O}(\delta\Theta^2). \tag{8}$$

If Θ_∞ corresponds to a stationary point of the potential, then

$$V'(\Theta_\infty) = 0, \tag{9}$$

and the leading-order equation becomes

$$\nabla^2\delta\Theta \approx V''(\Theta_\infty)\delta\Theta. \tag{10}$$

At sufficiently large r, the field variation becomes small and the dominant contribution is governed by the homogeneous equation

$$\nabla^2\delta\Theta \approx 0. \tag{11}$$

The general spherically symmetric solution of this equation is

$$\delta\Theta(r) = \frac{A}{r}, \tag{12}$$

where A is a constant determined by the strength of the localized source.

Thus, the scalar-time field generated by a compact, rotationally symmetric configuration exhibits the asymptotic behavior

$$\Theta_0(r) = \Theta_\infty + \frac{A}{r}. \tag{13}$$

This inverse-radial dependence arises directly from the scalar-time field equation and the requirement of spatial localization, and does not rely on any assumed interaction potential.

The background configuration $\Theta_0(r)$ defines a central field about which excitation modes are constructed in the following section.

3 Fluctuation Operator and Eigenvalue Problem

To determine the excitation spectrum supported by the scalar-time background configuration $\Theta_0(r)$, we consider perturbations of the form

$$\Theta(x,t) = \Theta_0(r) + \psi(x,t), \tag{14}$$

where $\psi(x,t)$ represents a small fluctuation about the static background.

Substituting into the scalar-time field equation

$$\Box\Theta = V'(\Theta), \tag{15}$$

and expanding to first order in ψ, we obtain the linearized equation

$$\Box\psi = V''\big(\Theta_0(r)\big)\psi. \tag{16}$$

For time-dependent fluctuations, we seek separable solutions of the form

$$\psi(x,t) = \psi(\mathbf{x})e^{-i\omega t}, \tag{17}$$

where ω is a real frequency parameter associated with the mode.

Substitution into the linearized equation yields

$$-\nabla^2\psi(\mathbf{x}) + V''\big(\Theta_0(r)\big)\psi(\mathbf{x}) = \omega^2\psi(\mathbf{x}). \tag{18}$$

This defines a linear operator

$$\mathcal{H}_\Theta = -\nabla^2 + V''\big(\Theta_0(r)\big), \tag{19}$$

such that the fluctuation modes satisfy

$$\mathcal{H}_\Theta\psi = \omega^2\psi. \tag{20}$$

The operator $\mathcal{H}_\Theta$ is real and symmetric under the standard inner product

$$\langle\psi_1,\psi_2\rangle = \int_{\mathbb{R}^3}\psi_1^*(\mathbf{x})\psi_2(\mathbf{x})\,d^3x. \tag{21}$$

Under appropriate boundary conditions ensuring finite energy and normalizability, the operator is self-adjoint. Its eigenfunctions form an orthonormal set, and its eigenvalues ω^2 are real.

Thus, the admissible fluctuation modes of the scalar-time field are given by the eigenfunctions of $\mathcal{H}_\Theta$, and the associated spectrum determines the allowed excitation energies.

The space of admissible states is therefore defined as the set of square-integrable eigenfunctions of $\mathcal{H}_\Theta$,

$$\mathcal{H} = \left\{\psi \in L^2(\mathbb{R}^3)\;\middle|\;\mathcal{H}_\Theta\psi = \omega^2\psi\right\}. \tag{22}$$

This space is complete under the inner product defined above, and provides the natural setting for describing scalar-time excitation modes.

No additional structure is imposed; the spectral properties arise directly from the scalar-time field equation and the background configuration $\Theta_0(r)$.

The determination of the spectrum therefore reduces to solving the eigenvalue problem defined by $\mathcal{H}_\Theta$ in the central background derived in the previous section.

4 Rotational Symmetry and Angular Momentum Structure

The background configuration derived above depends only on the radial coordinate,

$$\Theta_0(\mathbf{x}) = \Theta_0(r),$$

and is therefore invariant under spatial rotations. As a result, the fluctuation operator

$$\mathcal{H}_\Theta = -\nabla^2 + V''\big(\Theta_0(r)\big)$$

also possesses rotational symmetry.

Let L_i denote the generators of spatial rotations and

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

the total orbital angular momentum operator. Because the background depends only on r, the potential term commutes with rotations. Therefore,

$$[\mathcal{H}_\Theta, L^2] = 0, \tag{23}$$

and

$$[\mathcal{H}_\Theta, L_z] = 0. \tag{24}$$

Hence the eigenfunctions of $\mathcal{H}_\Theta$ may be chosen simultaneously as eigenfunctions of L^2 and L_z.

We therefore separate variables in spherical coordinates and write

$$\psi(\mathbf{x}) = R(r)Y_{\ell m}(\theta, \phi), \tag{25}$$

where $Y_{\ell m}$ are the angular eigenfunctions satisfying

$$L^2 Y_{\ell m} = \ell(\ell+1)Y_{\ell m}, \tag{26}$$

and

$$L_z Y_{\ell m} = m Y_{\ell m}. \tag{27}$$

The integers labeling these angular modes satisfy

$$\ell = 0, 1, 2, 3, \ldots \tag{28}$$

and

$$m = -\ell, -\ell+1, \ldots, \ell-1, \ell. \tag{29}$$

For each fixed value of ℓ, the number of allowed m values is

$$g_\ell = 2\ell + 1. \tag{30}$$

Thus the degeneracy of the scalar-time excitation spectrum associated with orbital angular structure arises directly from rotational symmetry of the background field.

Substituting the separated form into the eigenvalue equation

$$\mathcal{H}_\Theta \psi = \omega^2 \psi$$

shows that the angular dependence is completely determined by the spherical harmonics, while the radial dependence is governed by an ordinary differential equation whose form depends on the effective central potential $V''\big(\Theta_0(r)\big)$.

At this stage, the labels ℓ and m have not been assumed as quantum-mechanical postulates. They arise as a consequence of the rotational invariance of the scalar-time background and the corresponding symmetry of the fluctuation operator.

The excitation spectrum therefore decomposes into angular sectors indexed by ℓ, with multiplicity

$$2\ell + 1.$$

The next task is to derive the radial equation governing the bound-state structure within each angular sector.

5 Radial Equation and Effective Central Potential

We now derive the radial equation governing scalar-time excitation modes in each angular momentum sector.

Starting from the eigenvalue equation

$$\mathcal{H}_\Theta \psi = \omega^2 \psi,$$

with

$$\mathcal{H}_\Theta = -\nabla^2 + V''\big(\Theta_0(r)\big),$$

and using the separated form

$$\psi(\mathbf{x}) = R(r) Y_{\ell m}(\theta, \phi),$$

we substitute into the spectral equation.

In spherical coordinates, the Laplacian takes the form

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{L^2}{r^2}. \tag{31}$$

Applying this to the separated mode and using

$$L^2 Y_{\ell m} = \ell(\ell+1) Y_{\ell m},$$

we obtain

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) Y_{\ell m} + \frac{\ell(\ell+1)}{r^2} R Y_{\ell m} + V''\big(\Theta_0(r)\big) R Y_{\ell m} = \omega^2 R Y_{\ell m}. \tag{32}$$

Since the angular factor is nonzero and common to each term, it may be divided out, yielding the radial equation

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{\ell(\ell+1)}{r^2} R + V''\big(\Theta_0(r)\big) R = \omega^2 R. \tag{33}$$

This is the fundamental radial equation for scalar-time excitation modes in a static, rotationally symmetric background.

It is convenient to eliminate the first-derivative structure by defining the reduced radial function

$$u(r) = r R(r). \tag{34}$$

Substituting

$$R(r) = \frac{u(r)}{r}$$

into the radial equation gives

$$-\frac{d^2 u}{dr^2} + \left[\frac{\ell(\ell+1)}{r^2} + V''\big(\Theta_0(r)\big)\right] u = \omega^2 u. \tag{35}$$

Thus the radial problem takes the one-dimensional Sturm–Liouville form

$$-\frac{d^2u}{dr^2} + V_{\text{rad}}(r)u = \omega^2 u, \tag{36}$$

with effective radial potential

$$V_{\text{rad}}(r) = \frac{\ell(\ell+1)}{r^2} + V''\big(\Theta_0(r)\big). \tag{37}$$

The first term is the angular-curvature barrier arising from rotational symmetry. The second term is determined entirely by the scalar-time background field.

The structure of the bound-state spectrum is therefore controlled by two contributions:

1. the centrifugal term
$$\frac{\ell(\ell+1)}{r^2},$$
which increases with angular momentum,

2. the background-induced central term
$$V''\big(\Theta_0(r)\big),$$
which encodes the radial field structure of the localized source.

No external potential has been assumed. The effective central potential arises entirely from the second derivative of the scalar-time potential evaluated on the background solution $\Theta_0(r)$.

The existence of discrete bound states therefore depends on the asymptotic behavior of $V''\big(\Theta_0(r)\big)$. Since the background field was shown in the previous section to satisfy

$$\Theta_0(r) = \Theta_\infty + \frac{A}{r} \quad \text{for large } r,$$

the next task is to determine the corresponding asymptotic form of

$$V''\big(\Theta_0(r)\big),$$

and thereby derive the large-distance central potential governing atomic spectral structure.

6 Asymptotic Form of the Central Potential

The radial equation derived in Section 5 depends on the background-induced term

$$V''(\Theta_0(r)).$$

To determine the bound-state structure, we derive the asymptotic form of this quantity directly from the scalar-time background field.

From Section 2, the static, rotationally symmetric background satisfies

$$\Theta_0(r) = \Theta_\infty + \frac{A}{r},$$

for large r, where Θ_∞ is the asymptotic field value and A is a constant determined by the localized source.

We expand the second derivative of the scalar-time potential about Θ_∞. Since $\Theta_0(r) - \Theta_\infty = A/r$ becomes small at large r, we obtain

$$V''(\Theta_0(r)) = V''(\Theta_\infty) + V'''(\Theta_\infty)\,(\Theta_0(r) - \Theta_\infty) + O\big((\Theta_0 - \Theta_\infty)^2\big)\,.$$

Substituting the asymptotic form of $\Theta_0(r)$ yields

$$V''(\Theta_0(r)) = V''(\Theta_\infty) + \frac{A\,V'''(\Theta_\infty)}{r} + O\!\left(\frac{1}{r^2}\right).$$

The effective radial potential therefore becomes

$$V_{\mathrm{rad}}(r) = \frac{\ell(\ell+1)}{r^2} + V''(\Theta_\infty) + \frac{A\,V'''(\Theta_\infty)}{r} + O\!\left(\frac{1}{r^2}\right).$$

The constant term $V''(\Theta_\infty)$ sets the asymptotic mass scale of fluctuations. Since only energy differences are physically relevant, we absorb this constant into the spectral parameter by writing

$$\omega^2 = \omega_\infty^2 + \varepsilon, \qquad \omega_\infty^2 = V''(\Theta_\infty).$$

The radial equation then takes the asymptotic form

$$-\frac{d^2u}{dr^2} + \left[\frac{\ell(\ell+1)}{r^2} + \frac{A\,V'''(\Theta_\infty)}{r} + O\!\left(\frac{1}{r^2}\right)\right] u = \varepsilon\, u.$$

The leading long-range contribution is therefore inverse-radial:

$$V_{\mathrm{cent}}(r) \sim \frac{A\,V'''(\Theta_\infty)}{r}.$$

A discrete bound-state spectrum requires that this interaction be attractive. This condition is satisfied when

$$A\,V'''(\Theta_\infty) < 0.$$

This requirement is not imposed arbitrarily. It reflects a structural property of the scalar-time field: localized configurations corresponding to stable composite sources generate gradients that act as restoring forces for lower-energy excitations. In this sense, the attractive branch corresponds to configurations in which perturbations are drawn toward regions of maximal scalar-time coherence. This condition ensures that perturbations experience an attractive restoring gradient toward the localized source, consistent with the emergence of bound states.

When this condition is satisfied, we define a positive constant

$$\kappa = -A\,V'''(\Theta_\infty) > 0,$$

so that the asymptotic central potential takes the form

$$V_{\text{cent}}(r) \sim -\frac{\kappa}{r}.$$

Thus, the inverse-radial attractive interaction governing the bound-state spectrum is not assumed. It emerges from:

1. the scalar-time field equation,
2. the existence of a localized static background,
3. the asymptotic behavior $\Theta_0(r) = \Theta_\infty + A/r$,
4. the expansion of $V''(\Theta)$ about Θ_∞.

The resulting asymptotic radial equation is

$$-\frac{d^2u}{dr^2} + \left[\frac{\ell(\ell+1)}{r^2} - \frac{\kappa}{r}\right] u = \varepsilon\, u,$$

up to subleading corrections of order $1/r^2$.

This equation defines the universal large-distance bound-state problem generated by scalar-time field dynamics and determines the principal spectral structure derived in the following section.

7 Electrodynamic Normalization and Central Coupling

7.1 Emergent Electrodynamic Sector

In prior TSFT work, an effective electrodynamic structure emerges from the scalar-time field. In the static limit, the electric analogue is defined by

$$E_\Theta = -\nabla\Theta.$$

This identification corresponds to the interpretation of spatial gradients in the scalar-time field as effective force fields acting on perturbative excitations.

The inhomogeneous Maxwell equation is recovered in the form

$$\nabla \cdot E_\Theta = \frac{\rho_q}{\varepsilon_0},$$

together with the associated continuity equation

$$\partial_t \rho_q + \nabla \cdot J_q = 0,$$

ensuring conservation of charge.

Oppositely oriented scalar-time gradients correspond to opposite charge signs, leading to an attractive effective interaction for unlike charges, consistent with the binding condition derived in Section 6.

7.2 Gauss-Law Normalization

For a localized, spherically symmetric source, the asymptotic scalar-time field is

$$\Theta_0(r) = \Theta_\infty + \frac{A}{r}.$$

The corresponding field follows directly:

$$E_\Theta(r) = -\nabla\Theta_0(r) = \frac{A}{r^2}\,\hat{r}.$$

Integrating the Gauss-law equation over a ball B_R containing the source gives

$$\int_{B_R} \nabla \cdot E_\Theta \, d^3x = \frac{1}{\varepsilon_0}\int_{B_R} \rho_q \, d^3x.$$

Applying the divergence theorem yields

$$\oint_{\partial B_R} E_\Theta \cdot dS = \frac{Q_q}{\varepsilon_0},$$

where

$$Q_q := \int_{\mathbb{R}^3} \rho_q \, d^3x$$

is the total source charge.

Using the asymptotic form of $E_\Theta(r)$ gives

$$\oint_{\partial B_R} E_\Theta \cdot dS = 4\pi A,$$

so that

$$4\pi A = \frac{Q_q}{\varepsilon_0}, \qquad A = \frac{Q_q}{4\pi\varepsilon_0}.$$

Thus, the coefficient A is fixed by the conserved charge of the source.

7.3 Derivation of the Central Coupling

From Section 6, the asymptotic central interaction is

$$\kappa = -A\,V'''(\Theta_\infty).$$

Substituting the Gauss-law normalization gives

$$\kappa = -\frac{Q_q}{4\pi\varepsilon_0}\,V'''(\Theta_\infty).$$

For a localized source carrying charge number Z, we write

$$Q_q = Ze,$$

so that

$$\kappa = -\frac{Ze}{4\pi\varepsilon_0} V'''(\Theta_\infty).$$

Using the standard relation between e and the fine-structure constant,

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c},$$

we obtain

$$\kappa = -Z\frac{\alpha\hbar c}{e} V'''(\Theta_\infty).$$

Thus, the central coupling takes the form

$$\kappa = CZ\alpha,$$

with

$$C = -\frac{\hbar c}{e} V'''(\Theta_\infty).$$

7.4 Interpretation

The central coupling constant C is therefore not arbitrary. It is determined by:

1. the curvature of the scalar-time potential through $V'''(\Theta_\infty)$,
2. the normalization of the electrodynamic sector through Gauss-law structure.

The appearance of α reflects the inheritance of the electrodynamic normalization from the closure structure of TSFT. The quantity $V'''(\Theta_\infty)$ is fixed by the underlying scalar-time potential defining the theory, and therefore constitutes a structural parameter of the TSFT Lagrangian rather than an externally imposed constant.

Accordingly, the interaction strength governing atomic spectra is fixed by the chain

$$\Theta_0(r) \longrightarrow E_\Theta \longrightarrow Q_q \longrightarrow A \longrightarrow \kappa \longrightarrow C.$$

This establishes that both the form and normalization of the central interaction arise from scalar-time field dynamics and its emergent electrodynamic sector.

8 Discrete Bound States and Principal Spectral Structure

We now analyze the asymptotic radial equation obtained in the previous section:

$$-\frac{d^2u}{dr^2} + \left[\frac{\ell(\ell+1)}{r^2} - \frac{\kappa}{r}\right] u = \varepsilon\, u, \tag{38}$$

with $\kappa > 0$.

Bound states correspond to normalizable solutions. Therefore the spectral parameter must satisfy

$$\varepsilon < 0. \tag{39}$$

We write

$$\varepsilon = -\chi^2, \qquad \chi > 0, \tag{40}$$

so that the radial equation becomes

$$\frac{d^2u}{dr^2} + \left[-\chi^2 + \frac{\kappa}{r} - \frac{\ell(\ell+1)}{r^2}\right] u = 0. \tag{41}$$

To determine the admissible solutions, we examine the behavior at small and large r.

At large r, the $1/r$ and $1/r^2$ terms become subleading, and the equation reduces to

$$\frac{d^2u}{dr^2} - \chi^2 u = 0. \tag{42}$$

Its solutions are

$$u(r) \sim e^{\pm\chi r}. \tag{43}$$

Normalizability requires the decaying branch,

$$u(r) \sim e^{-\chi r} \qquad (r \to \infty). \tag{44}$$

At small r, the centrifugal term dominates. We therefore consider

$$\frac{d^2u}{dr^2} - \frac{\ell(\ell+1)}{r^2} u \approx 0. \tag{45}$$

The regular solution behaves as

$$u(r) \sim r^{\ell+1} \qquad (r \to 0). \tag{46}$$

These asymptotic behaviors motivate the ansatz

$$u(r) = r^{\ell+1} e^{-\chi r} f(r), \tag{47}$$

where $f(r)$ is assumed to be regular.

To simplify the equation, introduce the dimensionless variable

$$\rho = 2\chi r. \tag{48}$$

Then

$$u(r) = \rho^{\ell+1} e^{-\rho/2} f(\rho). \tag{49}$$

Substituting into the radial equation and simplifying yields

$$\rho \frac{d^2 f}{d\rho^2} + (2\ell + 2 - \rho)\frac{df}{d\rho} + \left(\frac{\kappa}{2\chi} - \ell - 1\right) f = 0. \tag{50}$$

This is the associated Laguerre equation. Its solutions are polynomials only when

$$\frac{\kappa}{2\chi} - \ell - 1 = n_r, \tag{51}$$

where

$$n_r = 0, 1, 2, 3, \ldots \tag{52}$$

is a nonnegative integer.

Therefore the admissible bound states satisfy

$$\chi = \frac{\kappa}{2(n_r + \ell + 1)}. \tag{53}$$

Defining the principal spectral index

$$n = n_r + \ell + 1, \qquad n = 1, 2, 3, \ldots, \tag{54}$$

we obtain

$$\chi = \frac{\kappa}{2n}. \tag{55}$$

Since $\varepsilon = -\chi^2$, the discrete bound-state spectrum is

$$\varepsilon_n = -\frac{\kappa^2}{4n^2}, \qquad n = 1, 2, 3, \ldots \tag{56}$$

Thus the scalar-time central potential derived in the previous section produces a discrete inverse-square spectral ladder indexed by the principal integer n.

The corresponding regular radial functions are

$$u_{n\ell}(r) = r^{\ell+1} e^{-\kappa r/(2n)} L_{n-\ell-1}^{2\ell+1}\left(\frac{\kappa r}{n}\right), \tag{57}$$

where

$$L_{n-\ell-1}^{2\ell+1} \tag{58}$$

denotes the associated Laguerre polynomial.

The full spatial excitation modes therefore take the form

$$\psi_{n\ell m}(r, \theta, \phi) = R_{n\ell}(r) Y_{\ell m}(\theta, \phi), \tag{59}$$

with

$$R_{n\ell}(r) = \frac{u_{n\ell}(r)}{r}. \tag{60}$$

At this stage, the principal spectral structure has been derived directly from the scalar-time field dynamics:

1. the scalar-time background produces an inverse-radial attractive potential,

2. rotational symmetry produces the angular sectors labeled by ℓ and m,

3. normalizability forces polynomial termination,

4. polynomial termination produces discrete values of n,
5. the spectrum scales as $-1/n^2$.

No Schrödinger equation or Coulomb potential was assumed independently. The discrete bound-state ladder arises as a consequence of the scalar-time field equation and the localized background derived earlier.

The remaining task is to determine the degeneracy of each principal level and derive the resulting shell capacities when fermionic spin and exclusion are included.

9 Degeneracy, Fermionic Spin, and Shell Capacities

We now determine the multiplicity of the bound states derived in the previous section and show how shell capacities emerge from symmetry and fermionic occupation.

For a fixed principal spectral index

$$n = n_r + \ell + 1,$$

the allowed orbital angular momentum values are determined by the condition

$$n_r \geq 0.$$

Since

$$n_r = n - \ell - 1,$$

this implies

$$\ell = 0, 1, 2, \ldots, n-1. \tag{61}$$

For each fixed ℓ, the magnetic quantum number takes the values

$$m = -\ell, -\ell+1, \ldots, \ell-1, \ell, \tag{62}$$

so the orbital degeneracy is

$$g_\ell = 2\ell + 1. \tag{63}$$

The total orbital degeneracy of a fixed principal level n is therefore

$$g_n^{(\mathrm{orb})} = \sum_{\ell=0}^{n-1} (2\ell + 1). \tag{64}$$

Evaluating the sum gives

$$g_n^{(\mathrm{orb})} = \sum_{\ell=0}^{n-1} 2\ell + \sum_{\ell=0}^{n-1} 1 = 2\frac{(n-1)n}{2} + n = n^2. \tag{65}$$

Thus the scalar-time spectral structure yields orbital degeneracy

$$g_n^{(\mathrm{orb})} = n^2. \tag{66}$$

We now incorporate fermionic spin. In TSFT, fermionic excitations carry intrinsic spin

$$s = \frac{1}{2},$$

as established previously from the spectral factorization structure of the scalar-time operator. For each spatial mode, there are therefore two independent spin states. In prior TSFT work, fermionic excitations arise from a first-order factorization of the scalar-time spectral operator, yielding spinor-valued solutions with a two-component internal structure (see [3], Papers 16–17). This enforces an intrinsic two-state multiplicity corresponding to spin $s = \frac{1}{2}$. The present work adopts this result without re-derivation.

Hence the total degeneracy of the principal level n is

$$g_n = 2\, g_n^{(\mathrm{orb})} = 2n^2. \tag{67}$$

This degeneracy determines the maximal number of fermions that can occupy the nth principal shell.

The shell capacities are therefore

$$g_1 = 2, \tag{68}$$

$$g_2 = 8, \tag{69}$$

$$g_3 = 18, \tag{70}$$

$$g_4 = 32, \tag{71}$$

and in general

$$g_n = 2n^2. \tag{72}$$

These capacities arise directly from:

1. the discrete scalar-time bound-state spectrum,
2. rotational symmetry,
3. the allowed range of orbital angular sectors for fixed n,
4. intrinsic fermionic spin,
5. Pauli exclusion.

No shell capacities have been assumed. They follow from the derived multiplicity of the scalar-time eigenmodes.

The resulting sequence

$$2,\ 8,\ 18,\ 32, \ldots \tag{73}$$

is the fundamental shell-capacity sequence associated with the principal spectral structure generated by the scalar-time field.

This provides the first direct connection between the scalar-time bound-state problem and atomic shell organization.

However, the principal-shell capacities alone do not yet explain the detailed ordering of subshells responsible for the full periodic structure of the elements. To reach that level, we must next analyze the internal organization of states within each principal shell and the role of angular sectors in determining spectral hierarchy.

10 Subshell Structure and Spectral Organization Within Principal Levels

The shell capacities derived above determine the total number of fermionic states in each principal spectral level. To move toward atomic periodicity, we must now resolve the internal organization of these states.

For fixed principal index n, the admissible orbital sectors are

$$\ell = 0, 1, 2, \ldots, n-1.$$

These correspond to distinct angular structures within the same principal level. Each such sector contributes degeneracy

$$2(2\ell + 1),$$

where the factor of 2 arises from spin.

Thus the decomposition of the nth principal shell is

$$2n^2 = \sum_{\ell=0}^{n-1} 2(2\ell + 1). \tag{74}$$

The first few shells therefore decompose as follows.

For $n = 1$,

$$2 = 2(1), \tag{75}$$

corresponding to the single sector

$$\ell = 0.$$

For $n = 2$,

$$8 = 2(1) + 2(3), \tag{76}$$

corresponding to

$$\ell = 0, 1.$$

For $n = 3$,

$$18 = 2(1) + 2(3) + 2(5), \tag{77}$$

corresponding to

$$\ell = 0, 1, 2.$$

For $n = 4$,

$$32 = 2(1) + 2(3) + 2(5) + 2(7), \tag{78}$$

corresponding to

$$\ell = 0, 1, 2, 3.$$

Thus the scalar-time spectral structure naturally produces subshell capacities

$$2,\ 6,\ 10,\ 14, \ldots \tag{79}$$

associated with the orbital sectors

$$\ell = 0, 1, 2, 3, \ldots$$

respectively.

These are the capacities conventionally associated with the

$$s,\ p,\ d,\ f,\ldots$$

subshells, but here they arise directly from rotational symmetry and fermionic spin, without being introduced as prior quantum-mechanical labels.

We now address the ordering of these sectors.

At the level of the asymptotic inverse-radial problem derived above, the energy depends only on the principal index n, so all sectors with the same n are degenerate:

$$\varepsilon_n = -\frac{\kappa^2}{4n^2}. \tag{80}$$

However, the full scalar-time background is not exhausted by its asymptotic form. The exact radial equation contains the complete central term

$$V''\big(\Theta_0(r)\big),$$

not merely its leading inverse-radial asymptotics. The subleading structure of this term lifts the exact degeneracy between sectors with different values of ℓ inside a fixed principal shell.

Accordingly, the exact bound-state energies must be written as

$$\varepsilon_{n\ell} = \varepsilon_n + \Delta_{n\ell}, \tag{81}$$

where $\Delta_{n\ell}$ denotes the correction produced by the inner structure of the scalar-time background.

The existence of these corrections follows directly from the fact that the exact radial potential is

$$V_{\text{rad}}(r) = \frac{\ell(\ell+1)}{r^2} + V''\big(\Theta_0(r)\big), \tag{82}$$

and the background field $\Theta_0(r)$ is not exactly equal to its asymptotic

$$\Theta_\infty + \frac{A}{r}$$

form at finite radius.

Therefore, while the leading principal-shell structure is determined by the asymptotic inverse-radial field, the detailed ordering of subshells is determined by the full central scalar-time background.

This gives the following structural hierarchy:

1. The asymptotic scalar-time field determines the principal-shell ladder

   $$2, 8, 18, 32, \ldots$$

   through the inverse-radial bound-state spectrum.

2. Rotational symmetry determines the internal subshell capacities

$$2, 6, 10, 14, \ldots$$

through the allowed orbital sectors.

3. The detailed ordering of subshell energies is fixed by the non-asymptotic inner structure of the scalar-time background.

The correction terms $\Delta_{n\ell}$ arise from deviations of the full scalar-time background $\Theta_0(r)$ from its asymptotic inverse-radial form. Explicitly, they are determined by solving the radial eigenvalue problem with the complete potential $V''(\Theta_0(r))$ rather than its leading-order expansion. In particular, the sign and relative magnitude of $\Delta_{n\ell}$ encode the transition from hydrogenic degeneracy to the structured subshell ordering observed in atomic systems.

Thus, the determination of subshell ordering is reduced to a well-posed spectral problem within TSFT: given a specific background configuration $\Theta_0(r)$, the shifts $\Delta_{n\ell}$ and their ordering follow uniquely from the corresponding Sturm–Liouville operator.

A full evaluation of $\Delta_{n\ell}$ for physically realistic source configurations is deferred to future work. Thus the scalar-time field yields both the shell capacities and the mechanism by which those shells are internally organized.

The remaining task is to state precisely what has been derived at this stage, and how this structure provides the necessary foundation for the later derivation of full elemental periodicity.

11 Derived Atomic Spectral Structure and Scope of Results

We now summarize the results obtained and clarify the scope of the derivation.

Starting from the scalar-time field action and its associated field equation, we constructed a static, spatially localized background configuration. Linearization about this background produced a self-adjoint fluctuation operator whose eigenmodes define the admissible excitation states.

From the asymptotic structure of the scalar-time background, we derived an effective inverse-radial central potential. The resulting radial equation admits normalizable solutions only for discrete values of the spectral parameter, yielding a bound-state spectrum of the form

$$\varepsilon_n = -\frac{\kappa^2}{4n^2}, \qquad n = 1, 2, 3, \ldots$$

Rotational symmetry of the background produces angular momentum sectors labeled by ℓ and m, with multiplicity

$$2\ell + 1.$$

The allowed range

$$\ell = 0, 1, 2, \ldots, n - 1$$

then yields total orbital degeneracy

$$n^2,$$

and inclusion of fermionic spin gives total level capacity

$$2n^2.$$

Thus the principal shell capacities

$$2,\ 8,\ 18,\ 32, \ldots$$

are obtained directly from the scalar-time spectral structure.

Within each principal shell, the decomposition into orbital sectors produces subshell capacities

$$2,\ 6,\ 10,\ 14, \ldots$$

which arise solely from rotational symmetry and fermionic spin.

At leading order, all states with the same n are degenerate. However, the exact scalar-time background deviates from its asymptotic inverse-radial form at finite radius. This produces corrections to the spectrum of the form

$$\varepsilon_{n\ell} = \varepsilon_n + \Delta_{n\ell},$$

which lift the degeneracy between different angular sectors and generate an internal ordering of subshells.

The derivation presented here establishes the following:

1. The existence of discrete atomic bound states follows from scalar-time field dynamics in a localized background.

2. The principal spectral ladder arises from the asymptotic inverse-radial structure of the scalar-time field.

3. Shell capacities follow from degeneracy determined by rotational symmetry and fermionic spin.

4. Subshell structure arises from the decomposition of angular momentum sectors.

5. Subshell ordering is determined by the detailed radial structure of the scalar-time background.

No external central potential, wave equation, or quantum-mechanical postulate has been introduced. All structure follows from the scalar-time field equation and its symmetries.

The results obtained here provide the necessary spectral foundation for atomic structure. However, they do not yet determine the detailed ordering of subshell energies required to reproduce the full periodic behavior of elements. That ordering depends on the non-asymptotic structure of the scalar-time background and the interactions between multiple fermions occupying these states.

The extension of this framework to full periodicity therefore requires analysis of multi-particle occupation and interaction-induced spectral shifts, which lie beyond the scope of the present work.

12 Discussion

The derivation presented in this work establishes atomic spectral structure as a direct consequence of scalar-time field dynamics. It is therefore useful to examine the physical content of the result and its relation to existing descriptions.

The central outcome is that a localized scalar-time configuration generates an inverse-radial background field at large distances. This asymptotic behavior leads to a discrete bound-state spectrum with principal index n, angular sectors labeled by ℓ and m, and degeneracy structure determined entirely by symmetry and fermionic occupation.

In conventional formulations, this same structure is obtained by solving the Schrödinger or Dirac equation in a Coulomb potential. In the present derivation, the inverse-radial form of the potential is not assumed. It arises from the scalar-time field equation together with the requirement of spatial localization. Thus the central binding structure is a consequence of the field dynamics rather than an external input.

The appearance of angular momentum sectors also follows directly from symmetry. Rotational invariance of the background field enforces conservation of angular momentum and produces the decomposition of the spectrum into sectors with multiplicity $2\ell + 1$. The resulting degeneracy structure is therefore not introduced as a quantum-mechanical postulate but emerges from the invariance properties of the scalar-time configuration.

The principal-shell degeneracy $2n^2$ and the subshell capacities $2(2\ell + 1)$ then follow as algebraic consequences of the allowed angular sectors and the presence of intrinsic fermionic spin. The shell capacities

$$2, \ 8, \ 18, \ 32, \ldots$$

are therefore determined by the spectral geometry of the scalar-time operator.

A further important point is the role of the asymptotic approximation. The inverse-radial potential governs the large-distance behavior of the system and determines the existence of discrete bound states and their principal-level organization. However, the detailed structure of the scalar-time background at finite radius introduces corrections that distinguish between different angular sectors within a given principal shell.

These corrections are not arbitrary. They arise from the exact radial dependence of $\Theta_0(r)$, which is fixed by the scalar-time field equation and the internal structure of the localized source. As a result, the ordering of subshell energies is determined by the detailed geometry of the scalar-time background rather than by externally imposed rules.

It is important to emphasize the scope of the present derivation. The results obtained here establish the existence of atomic shells and their capacities, but do not yet determine the full sequence of subshell energy ordering required to reproduce the detailed periodic structure of the elements. That ordering depends on additional effects, including the precise radial profile of the scalar-time background and interactions between multiple fermionic excitations.

Nevertheless, the present framework provides the necessary foundation for such extensions. By deriving the principal spectral structure and its degeneracies directly from the scalar-time field, it reduces the problem of periodicity to a well-defined question: the determination of how the detailed scalar-time background and multi-particle effects lift the degeneracy within each principal shell.

In this sense, atomic spectral structure appears as an intermediate layer in a hierarchy of scalar-time organization, linking the previously derived particle and nuclear structures to the emergent periodic behavior of chemical elements.

13 Conclusion

In this work, we have derived the structural form of atomic bound-state spectra from Time-Scalar Field Theory (TSFT) without postulating quantum mechanics or introducing a Coulomb potential. Starting from the scalar-time field equation, we constructed localized, static background solutions with asymptotic behavior

$$\Theta_0(r) = \Theta_\infty + \frac{A}{r}.$$

Expanding the induced operator $V''(\Theta_0(r))$ about Θ_∞ yields an emergent inverse-radial interaction at large distances. Requiring the existence of normalizable bound states selects the attractive branch and defines the central coupling

$$V_{\text{cent}}(r) \sim -\frac{\kappa}{r},$$

with κ determined by the background amplitude and the curvature of the scalar-time potential.

The absolute normalization of this interaction is fixed by the recovered electrodynamic sector of TSFT. Using $E_\Theta = -\nabla\Theta$ together with the inhomogeneous Maxwell equation $\nabla \cdot E_\Theta = \rho_q/\varepsilon_0$, Gauss-law normalization relates the asymptotic coefficient A to the conserved source charge. This yields

$$\kappa = CZ\alpha,$$

where Z is the source charge number, α is the fine-structure constant inherited from the closure structure of TSFT, and C is determined by the scalar-time potential curvature and electrodynamic normalization. No Coulomb potential has been independently assumed; both the form and normalization of the interaction arise from scalar-time field dynamics and its emergent electrodynamic sector.

The resulting radial equation defines a Sturm–Liouville problem with inverse-radial leading behavior. Imposing normalizability yields Laguerre termination and a discrete spectrum

$$\varepsilon_n = -\frac{\kappa^2}{4n^2},$$

together with shell capacities $2n^2$. Subleading corrections of order $1/r^2$ generate subshell-dependent shifts $\Delta_{n\ell}$, which are treated perturbatively and encode the detailed structure of atomic levels.

Accordingly, the principal organization of atomic spectra emerges from the combined effects of scalar-time field structure, spatial localization, and electrodynamic normalization. The derivation is non-circular: the inverse-radial interaction, spectral quantization, and shell structure follow directly from the scalar-time framework without importing the standard quantum-mechanical formalism.

The remaining problem is the explicit evaluation of the subshell shifts $\Delta_{n\ell}$ for realistic source configurations. This requires solving the full radial operator with the complete background field $\Theta_0(r)$ and will determine the detailed ordering of atomic levels. Addressing this problem constitutes the next step toward a fully predictive account of atomic structure within TSFT.

In summary, the present work establishes that the structural form of atomic spectra, including the hydrogenic energy ladder and shell capacities, is a consequence of scalar-time field dynamics together with the internally recovered electrodynamic sector. This provides a coherent and mathematically controlled route from first principles to atomic spectral organization.

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A Recovery of Hydrogen Orbital Structure

In this appendix we demonstrate that the scalar-time spectral structure derived in the main text reproduces the spatial structure of atomic orbitals as a direct consequence of the eigenfunctions of the fluctuation operator.

A.1 Eigenfunction Structure

From the radial and angular separation derived in Sections 5 and 6, the bound-state eigenfunctions take the form

$$\psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)Y_{\ell m}(\theta,\phi), \tag{83}$$

where $Y_{\ell m}$ are the angular eigenfunctions determined by rotational symmetry, and $R_{n\ell}(r)$ is the radial function.

Using the result obtained in Section 6, the radial functions are given by

$$R_{n\ell}(r) = \frac{1}{r} r^{\ell+1} e^{-\kappa r/(2n)} L_{n-\ell-1}^{2\ell+1}\left(\frac{\kappa r}{n}\right), \tag{84}$$

which simplifies to

$$R_{n\ell}(r) = r^{\ell} e^{-\kappa r/(2n)} L_{n-\ell-1}^{2\ell+1}\left(\frac{\kappa r}{n}\right). \tag{85}$$

Thus the full eigenfunctions are

$$\psi_{n\ell m}(r,\theta,\phi) = r^{\ell} e^{-\kappa r/(2n)} L_{n-\ell-1}^{2\ell+1}\left(\frac{\kappa r}{n}\right) Y_{\ell m}(\theta,\phi). \tag{86}$$

A.2 Probability Density

The observable spatial distribution associated with each mode is given by the squared magnitude

$$\rho_{n\ell m}(r,\theta,\phi) = |\psi_{n\ell m}(r,\theta,\phi)|^2. \tag{87}$$

This yields

$$\rho_{n\ell m} = r^{2\ell} e^{-\kappa r/n} \left[L_{n-\ell-1}^{2\ell+1}\left(\frac{\kappa r}{n}\right)\right]^2 |Y_{\ell m}(\theta,\phi)|^2. \tag{88}$$

The spatial structure therefore factorizes into:

1. a radial component determined by the scalar-time background,
2. an angular component determined by rotational symmetry.

A.3 Nodal Structure

The structure of the eigenfunctions is governed by two types of nodes:

Radial nodes The associated Laguerre polynomial

$$L_{n-\ell-1}^{2\ell+1}$$

has degree $n - \ell - 1$, and therefore produces

$$n - \ell - 1 \tag{89}$$

radial nodes.

Angular nodes The angular eigenfunctions $Y_{\ell m}$ exhibit nodal surfaces determined by ℓ and m, producing:

$$\ell \quad \text{angular nodal surfaces.} \tag{90}$$

Thus the total nodal structure is fully determined by the integers (n, ℓ, m).

A.4 Orbital Geometry

The spatial patterns associated with the lowest angular sectors are:

- $\ell = 0$: spherically symmetric distributions,
- $\ell = 1$: dipolar distributions with one nodal plane,
- $\ell = 2$: quadrupolar distributions with multiple nodal surfaces,
- $\ell = 3$: higher-order multipolar structures.

These geometries arise directly from the angular eigenfunctions and are not introduced independently.

A.5 Interpretation

The results above show that the spatial structure of atomic orbitals is a direct consequence of the scalar-time eigenvalue problem derived in the main text.

Specifically:

1. The inverse-radial scalar-time background determines the radial envelope of the eigenfunctions.
2. Rotational symmetry determines the angular structure and associated nodal surfaces.
3. Normalizability enforces polynomial termination, producing discrete radial modes.
4. The combination of these effects yields the full spatial structure of bound states.

Thus the characteristic orbital shapes are not assumed, but emerge from the geometry of scalar-time excitation modes.

A.6 Summary

The scalar-time field framework reproduces:

- discrete bound-state spectrum,
- angular momentum structure,
- radial and angular nodal patterns,
- spatial orbital geometry.

These results establish that the observed structure of atomic orbitals is an intrinsic consequence of scalar-time spectral dynamics.

B Orthogonality, Completeness, and Normalization of Eigenmodes

In this appendix we establish the orthogonality, completeness, and normalization properties of the scalar-time eigenfunctions derived in the main text.

B.1 Self-Adjoint Structure

The fluctuation operator derived in Section 4 is

$$\mathcal{H}_\Theta = -\nabla^2 + V''\big(\Theta_0(r)\big). \tag{91}$$

Under the inner product

$$\langle \psi_1, \psi_2 \rangle = \int_{\mathbb{R}^3} \psi_1^*(\mathbf{x})\psi_2(\mathbf{x})\, d^3x, \tag{92}$$

and with boundary conditions ensuring square integrability, the operator $\mathcal{H}_\Theta$ is self-adjoint.

Therefore, its eigenvalues are real and its eigenfunctions corresponding to distinct eigenvalues are orthogonal.

B.2 Angular Orthogonality

The angular functions satisfy

$$\int Y_{\ell m}^*(\theta,\phi)\, Y_{\ell' m'}(\theta,\phi)\, d\Omega = \delta_{\ell\ell'}\delta_{mm'}, \tag{93}$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$.

Thus the angular sectors are orthogonal.

B.3 Radial Orthogonality

The radial equation takes the Sturm–Liouville form

$$-\frac{d^2u}{dr^2} + V_{\mathrm{rad}}(r)u = \varepsilon u, \tag{94}$$

with

$$V_{\mathrm{rad}}(r) = \frac{\ell(\ell+1)}{r^2} + V''\big(\Theta_0(r)\big). \tag{95}$$

This operator is self-adjoint under the measure dr, so radial eigenfunctions with distinct eigenvalues satisfy

$$\int_0^\infty u_{n\ell}(r)\, u_{n'\ell}(r)\, dr = 0 \quad \text{for } n \neq n'. \tag{96}$$

Equivalently, for the full radial functions,

$$\int_0^\infty R_{n\ell}(r)\, R_{n'\ell}(r)\, r^2 dr = 0 \quad \text{for } n \neq n'. \tag{97}$$

B.4 Full Eigenfunction Orthogonality

Combining radial and angular parts, the full eigenfunctions satisfy

$$\int \psi^*_{n\ell m}(\mathbf{x})\psi_{n'\ell' m'}(\mathbf{x})\, d^3x = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}. \tag{98}$$

Thus the eigenmodes form an orthogonal set.

B.5 Normalization

Each eigenfunction may be normalized so that

$$\int |\psi_{n\ell m}(\mathbf{x})|^2\, d^3x = 1. \tag{99}$$

This fixes the overall normalization constant of the radial function $R_{n\ell}(r)$.

B.6 Completeness

Because $\mathcal{H}_\Theta$ is self-adjoint on a Hilbert space of square-integrable functions, its eigenfunctions form a complete basis.

Thus any admissible scalar-time fluctuation $\psi(\mathbf{x})$ may be expanded as

$$\psi(\mathbf{x}) = \sum_{n,\ell,m} c_{n\ell m}\, \psi_{n\ell m}(\mathbf{x}), \tag{100}$$

with coefficients

$$c_{n\ell m} = \int \psi^*_{n\ell m}(\mathbf{x})\, \psi(\mathbf{x})\, d^3x. \tag{101}$$

B.7 Interpretation

These results establish that:

1. The scalar-time eigenmodes form a complete orthonormal basis.
2. All admissible excitation states can be expressed as superpositions of these modes.
3. The spectral decomposition is entirely determined by the scalar-time operator $\mathcal{H}_\Theta$.

No independent Hilbert space structure has been assumed. The space of states arises as the natural consequence of the spectral properties of the scalar-time field operator.

C Radial Distributions and Expectation Values

In this appendix we compute expectation values associated with the scalar-time eigenmodes and analyze the radial structure of bound states.

C.1 Radial Probability Density

The probability density is given by

$$\rho_{n\ell m}(r,\theta,\phi) = |\psi_{n\ell m}(r,\theta,\phi)|^2. \tag{102}$$

Integrating over angular variables yields the radial probability density

$$P_{n\ell}(r) = \int \rho_{n\ell m}(r,\theta,\phi)\, d\Omega = |R_{n\ell}(r)|^2 r^2. \tag{103}$$

Thus the radial distribution depends only on n and ℓ.

C.2 Normalization Condition

The normalization condition becomes

$$\int_0^\infty P_{n\ell}(r)\, dr = 1. \tag{104}$$

C.3 Expectation Value of Radius

The expectation value of the radial coordinate is

$$\langle r\rangle_{n\ell} = \int_0^\infty r\, P_{n\ell}(r)\, dr. \tag{105}$$

Using the form of the radial functions,

$$R_{n\ell}(r) = r^\ell e^{-\kappa r/(2n)} L^{2\ell+1}_{n-\ell-1}\left(\frac{\kappa r}{n}\right), \tag{106}$$

the dominant contribution to the integral arises from the region

$$r \sim \frac{2n}{\kappa}. \tag{107}$$

Thus the expectation value scales as

$$\langle r \rangle_{n\ell} \propto \frac{n^2}{\kappa}. \tag{108}$$

C.4 Radial Scaling Law

More generally, dimensional analysis of the radial equation

$$-\frac{d^2u}{dr^2} + \left[\frac{\ell(\ell+1)}{r^2} - \frac{\kappa}{r}\right] u = \varepsilon u \tag{109}$$

shows that the characteristic length scale is

$$r_n \sim \frac{n^2}{\kappa}. \tag{110}$$

Thus higher principal levels correspond to increasingly extended spatial distributions.

C.5 Radial Probability Peaks

The radial probability density

$$P_{n\ell}(r) = |R_{n\ell}(r)|^2 r^2 \tag{111}$$

exhibits peaks determined by the interplay between:

- exponential decay $e^{-\kappa r/n}$,
- polynomial growth from Laguerre functions,
- centrifugal suppression at small r.

The number of peaks is determined by the number of radial nodes:

$$n - \ell - 1. \tag{112}$$

C.6 Angular Independence of Radial Observables

Since the angular functions are normalized independently,

$$\int |Y_{\ell m}|^2 d\Omega = 1, \tag{113}$$

radial expectation values are independent of m.

Thus all states with the same (n, ℓ) share identical radial distributions.

C.7 Energy-Length Relationship

Using the spectral result

$$\varepsilon_n = -\frac{\kappa^2}{4n^2}, \tag{114}$$

and the scaling

$$r_n \sim \frac{n^2}{\kappa}, \tag{115}$$

we obtain

$$\varepsilon_n \sim -\frac{1}{r_n}. \tag{116}$$

Thus the binding energy is inversely proportional to the characteristic size of the bound state.

C.8 Interpretation

These results show that:

1. Bound states exhibit a hierarchical spatial structure controlled by n.
2. The size of orbitals grows quadratically with the principal index.
3. Radial node structure is determined by polynomial termination.
4. Energy and spatial scale are inversely related.

All of these properties arise directly from the scalar-time field dynamics and the derived inverse-radial central potential.

D Appendix D: Subleading Radial Corrections and Subshell Structure

D.1 Expansion of the Effective Potential

In the main text, the leading-order behavior of the effective radial potential arises from the asymptotic expansion

$$V''(\Theta_0(r)) \sim -\frac{\kappa}{r},$$

which yields the inverse-radial form responsible for the principal spectral structure.

At higher order, the expansion of $V''(\Theta_0(r))$ generates additional terms, including a subleading contribution of the form

$$V_{\text{corr}}(r) = \frac{\beta}{r^2},$$

where β is determined by higher derivatives of the scalar-time potential evaluated about Θ_∞.

D.2 Perturbative Regime

For physically relevant configurations, the subleading term is expected to remain small relative to the centrifugal barrier,

$$\frac{\ell(\ell+1)}{r^2}.$$

Accordingly, we restrict to the perturbative regime

$$|\beta| \ll \ell(\ell+1),$$

in which the inverse-radial structure derived in the main text remains dominant.

In this regime, the β/r^2 contribution may be treated as a perturbation to the leading Sturm–Liouville problem.

D.3 Energy Corrections

The unperturbed eigenfunctions $\psi_{n\ell}$ obtained from the inverse-radial problem provide a complete basis for computing the effect of the subleading term.

To first order in perturbation theory, the energy correction is

$$\Delta_{n\ell} = \left\langle \psi_{n\ell} \left| \frac{\beta}{r^2} \right| \psi_{n\ell} \right\rangle.$$

For bound states of the inverse-radial problem, the expectation value satisfies

$$\left\langle \frac{1}{r^2} \right\rangle \sim \frac{\kappa^2}{n^3},$$

so that

$$\Delta_{n\ell} \propto \frac{\beta}{n^3}.$$

Thus, the corrected spectrum takes the form

$$\varepsilon_{n\ell} = -\frac{\kappa^2}{4n^2} + \Delta_{n\ell},$$

with the leading structure preserved and subshell-dependent shifts arising from the perturbation.

D.4 Interpretation

The β/r^2 term represents the first correction to the idealized inverse-radial potential and encodes deviations of the full scalar-time background $\Theta_0(r)$ from its asymptotic form.

These corrections lift the degeneracy of states with fixed n but different ℓ, thereby generating subshell structure.

The determination of the detailed ordering of subshell energies is therefore reduced to the evaluation of $\Delta_{n\ell}$ for physically realistic source configurations, which is deferred to future work.

D.5 Consistency with the Main Derivation

Importantly, the perturbative treatment preserves the exact solvability of the leading-order problem and avoids modifying the angular momentum structure.

The principal spectral result

$$\varepsilon_n = -\frac{\kappa^2}{4n^2}$$

remains exact at leading order, while all subshell structure arises from controlled corrections.

This ensures that the derivation of atomic spectral organization from the scalar-time field remains non-circular and mathematically consistent.

E Predictive Consequences and Testable Relations

The scalar-time derivation of atomic spectral structure yields a set of parameter-independent relations that can be tested against experiment. These follow directly from the structure established in the main text and do not introduce additional assumptions.

E.1 Subshell Splitting Scaling

The energy corrections arising from the full scalar-time background produce subshell splittings within each principal level,

$$\varepsilon_{n\ell} = \varepsilon_n + \Delta_{n\ell}.$$

From the radial scaling

$$r_n \sim \frac{n^2}{\kappa},$$

and the fact that corrections arise from higher-order terms in

$$V''\big(\Theta_0(r)\big),$$

the magnitude of these splittings satisfies the scaling relation

$$\Delta_{n\ell} \propto \frac{1}{n^3}, \tag{117}$$

up to slowly varying factors determined by the detailed radial structure.

Thus, subshell splittings decrease with increasing principal index.

E.2 Degeneracy Constraints

Rotational symmetry of the scalar-time background enforces

$$\varepsilon_{n\ell m} = \varepsilon_{n\ell}, \tag{118}$$

so that all states with different m remain degenerate.

Therefore, any intrinsic splitting of energy levels must depend only on ℓ and not on m.

E.3 Radial Scaling of Bound States

The characteristic size of bound states satisfies

$$r_n \sim \frac{n^2}{\kappa}. \tag{119}$$

Combined with the energy spectrum

$$\varepsilon_n = -\frac{\kappa^2}{4n^2}, \tag{120}$$

this yields the relation

$$|\varepsilon_n|\, r_n \sim \kappa, \tag{121}$$

up to numerical factors.

Thus, the binding energy and spatial extent of bound states are inversely related.

E.4 Radial Node Structure

The number of radial nodes is

$$n - \ell - 1. \tag{122}$$

The positions of these nodes are determined by the zeros of the associated Laguerre polynomials and therefore satisfy universal dimensionless relations

$$\frac{r_k}{r_n} = x_k(n, \ell), \tag{123}$$

independent of the specific scalar-time background scale.

E.5 Large-n Limit

For large principal index n, the correction terms become small relative to the leading spectrum:

$$\frac{\Delta_{n\ell}}{\varepsilon_n} \to 0. \tag{124}$$

Thus the spectrum approaches degeneracy within each principal level, and subshell distinctions become negligible.

E.6 Subshell Ordering from Background Structure

The ordering of subshell energies is determined by the full radial dependence of the scalar-time background through

$$V''\big(\Theta_0(r)\big).$$

Thus, for any specified background configuration, the subshell ordering is uniquely determined without the introduction of empirical ordering rules.

E.7 Summary

The scalar-time framework yields the following testable predictions:

1. Subshell splittings decrease approximately as n^{-3}.
2. Intrinsic level splitting depends only on ℓ and not on m.
3. Bound-state size scales as n^2.
4. Binding energy scales as n^{-2}.
5. Energy and spatial scale satisfy an inverse relationship.
6. Radial node positions obey universal dimensionless ratios.
7. Subshell ordering is determined by the scalar-time background rather than empirical rules.

These relations provide a set of constraints that can be compared directly with experimental atomic spectra and spatial distributions.

F Structural Lemmas of Scalar-Time Atomic Spectra

In this appendix we summarize the principal structural results derived in the main text in the form of lemmas. Each lemma follows directly from the scalar-time field equation, the construction of the central background, and the spectral analysis of the fluctuation operator.

Lemma 1 (Principal Spectrum)

For a scalar-time background with asymptotic form

$$\Theta_0(r) = \Theta_\infty + \frac{A}{r},$$

the fluctuation operator admits discrete bound states with energies

$$\varepsilon_n = -\frac{\kappa^2}{4n^2}, \qquad n = 1, 2, 3, \ldots, \tag{125}$$

where $\kappa = -AV'''(\Theta_\infty) > 0$.

Lemma 2 (Angular Degeneracy)

Rotational symmetry of the scalar-time background implies that the eigenfunctions decompose into angular sectors labeled by ℓ and m, with multiplicity

$$g_\ell = 2\ell + 1. \tag{126}$$

Lemma 3 (Principal-Level Degeneracy)

For each principal index n, the allowed angular sectors satisfy

$$\ell = 0, 1, \ldots, n-1,$$

and the total orbital degeneracy is

$$g_n^{(\text{orb})} = n^2. \tag{127}$$

Lemma 4 (Shell Capacity)

Including fermionic spin $s = \frac{1}{2}$, each spatial mode admits two independent spin states. Therefore, the total capacity of the nth principal level is

$$g_n = 2n^2. \tag{128}$$

Lemma 5 (Subshell Structure)

Each principal level decomposes into subshells labeled by ℓ with capacities

$$g_\ell^{(\text{sub})} = 2(2\ell + 1), \tag{129}$$

corresponding to the angular sectors.

Lemma 6 (Radial Node Structure)

The radial eigenfunctions contain

$$n - \ell - 1 \tag{130}$$

radial nodes, determined by the degree of the associated Laguerre polynomials.

Lemma 7 (Radial Scaling)

The characteristic radial scale of bound states satisfies

$$r_n \sim \frac{n^2}{\kappa}. \tag{131}$$

Lemma 8 (Energy–Length Relation)

The bound-state energy and spatial scale satisfy the relation

$$|\varepsilon_n|\, r_n \sim \kappa, \tag{132}$$

up to numerical factors.

Lemma 9 (Degeneracy Lifting)

Corrections arising from the full scalar-time background produce ℓ-dependent energy shifts

$$\varepsilon_{n\ell} = \varepsilon_n + \Delta_{n\ell}, \tag{133}$$

which lift the degeneracy within each principal level.

Lemma 10 (Subshell Ordering)

The ordering of subshell energies is uniquely determined by the radial dependence of the scalar-time background through

$$V''\big(\Theta_0(r)\big),$$

and does not require independent empirical rules.

Lemma 11 (Asymptotic Degeneracy)

In the limit $n \to \infty$, the correction terms satisfy

$$\frac{\Delta_{n\ell}}{\varepsilon_n} \to 0, \tag{134}$$

so that all states within a principal level become asymptotically degenerate.

Summary

These lemmas collectively establish that the principal features of atomic spectral structure—discrete bound states, shell capacities, subshell decomposition, and degeneracy structure—arise as direct consequences of scalar-time field dynamics.

G Appendix G: Closure Structure and Electrodynamic Consistency

G.1 Closure Residue and the Fine-Structure Constant

In prior TSFT work, the fine-structure constant α arises as a closure residue associated with repeated projection of scalar-time modes. Specifically, spectral closure requires that phase evolution return to itself after a finite number of cycles, yielding the condition

$$\lambda^N = 1,$$

for eigenvalue λ on the unit circle.

This implies a discrete structure for the effective coupling parameter, which may be written in the form

$$\alpha = \frac{1}{N},$$

for some integer closure index N. The empirically observed value $\alpha \approx 1/137$ corresponds to the minimal stable closure branch identified in the TSFT spectral construction (see Vol. 3, Appendix M).

The present work does not re-derive this selection, but instead inherits the value of α from the established closure structure.

G.2 Gauss-Law Normalization of the Scalar-Time Field

The electrodynamic sector of TSFT provides a normalization condition relating the scalar-time field to physical charge. In the static limit, the field satisfies

$$E_\Theta = -\nabla\Theta,$$

together with the inhomogeneous Maxwell equation

$$\nabla \cdot E_\Theta = \frac{\rho_q}{\varepsilon_0}.$$

Integration over space yields the Gauss-law relation

$$\oint E_\Theta \cdot dS = \frac{Q_q}{\varepsilon_0},$$

which fixes the asymptotic behavior of the scalar-time field as

$$\Theta_0(r) = \Theta_\infty + \frac{A}{r}, \qquad A = \frac{Q_q}{4\pi\varepsilon_0}.$$

Thus, the far-field coefficient A is determined by the conserved charge of the source, rather than introduced phenomenologically.

G.3 Derivation of the Central Coupling

From Section 6, the asymptotic central interaction is

$$\kappa = -A\, V'''(\Theta_\infty).$$

Substituting the Gauss-law normalization gives

$$\kappa = -\frac{Q_q}{4\pi\varepsilon_0}\, V'''(\Theta_\infty).$$

For a localized source of charge number Z, with $Q_q = Ze$, this becomes

$$\kappa = -\frac{Ze}{4\pi\varepsilon_0}\, V'''(\Theta_\infty).$$

Using the standard relation between e and α,

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c},$$

we obtain

$$\kappa = -Z\,\frac{\alpha\hbar c}{e}\,V'''(\Theta_\infty).$$

Thus, the central coupling takes the form

$$\kappa = CZ\alpha,$$

with

$$C = -\frac{\hbar c}{e}\,V'''(\Theta_\infty).$$

G.4 Consistency with the Closure Structure

The appearance of α in the central coupling is therefore not ad hoc, but reflects the inheritance of the electrodynamic normalization from the scalar-time closure structure.

The atomic spectral problem depends on κ, and hence on α, through two independent but consistent mechanisms:

1. The Gauss-law normalization of the scalar-time field, which fixes the relation between source charge and field amplitude.
2. The closure structure of TSFT, which determines the value of the fine-structure constant as a spectral residue.

Together, these establish that the coupling strength entering the atomic spectrum is a direct consequence of the scalar-time field structure, rather than an externally imposed parameter.

G.5 Summary

The normalization chain derived in the present work is

$$\Theta_0(r) \;\longrightarrow\; E_\Theta \;\longrightarrow\; Q_q \;\longrightarrow\; A \;\longrightarrow\; \kappa \;\longrightarrow\; C.$$

This chain demonstrates that the atomic coupling constant is fixed by the combination of scalar-time potential curvature and electrodynamic normalization, with α entering as a closure-determined parameter of the theory.