A Spectral Closure Operator in Time–Scalar Field Theory and the Riemann Hypothesis

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The Riemann Hypothesis (RH) asserts that every nontrivial zero of the Riemann zeta function $\zeta(s)$ lies on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. A longstanding program initiated by Hilbert and Pólya suggests that RH would follow if the imaginary parts of the nontrivial zeros correspond to the spectrum of a self-adjoint operator.

In this work we construct a spectral operator within the framework of Time–Scalar Field Theory (TSFT), where physical and mathematical structures arise as coherent eigenmodes of a scalar-time field. We show that under natural TSFT closure conditions the spectral determinant of the resulting operator corresponds to the completed Riemann function $\xi(s)$. If the operator is shown to be self-adjoint on a suitable domain, its eigenvalues are real, implying that the admissible zeros of $\xi(s)$ must lie on the critical line.

The construction is motivated by three independent structures within the TSFT program: (i) spectral projection operators governing coherent temporal modes, (ii) prime-power weighting arising from irreducible harmonic structure, and (iii) logarithmic localization consistent with the explicit formula for prime distributions. These ingredients lead naturally to a logarithmic dilation operator supplemented by a scalar-time closure potential.

The resulting framework provides a candidate Hilbert–Pólya realization emerging from TSFT spectral geometry. Supporting analysis using Gaussian witness probes is discussed as a consistency check on off-critical instability.

I. INTRODUCTION

The Riemann Hypothesis (RH) is one of the central unsolved problems of modern mathematics. It states that all nontrivial zeros of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \qquad \mathrm{Re}(s) > 1, \tag{1}$$

extend analytically to the complex plane with the property that every nontrivial zero satisfies

$$\rho = \frac{1}{2} + i\gamma. \tag{2}$$

Despite extensive numerical verification and numerous partial results, a complete proof remains elusive. Many approaches attack RH through analytic inequalities or estimates derived from the explicit formula connecting zeros of $\zeta(s)$ to the distribution of prime numbers. While these methods have produced deep insights into number theory, they have not yet produced a decisive argument establishing the critical-line condition.

A conceptually different program was proposed by Hilbert and Pólya: the possibility that the imaginary parts of the nontrivial zeros arise as eigenvalues of a self-adjoint operator. If such an operator $\mathcal{H}$ existed with spectrum

$$\mathcal{H}\psi_n = \gamma_n \psi_n, \tag{3}$$

then the reality of the eigenvalues γ_n would imply

$$\rho_n = \frac{1}{2} + i\gamma_n, \tag{4}$$

establishing RH.

Over the past several decades, various physical and mathematical candidates for such an operator have been proposed. These include Berry–Keating type Hamiltonians associated with logarithmic dilation generators, as well as non-Hermitian constructions whose spectra are conjectured to reproduce the Riemann zeros. However, a fully satisfactory realization of a self-adjoint operator whose spectral determinant reproduces the Riemann ξ-function remains unknown.

In this paper we explore a new realization of the Hilbert–Pólya program arising from the framework of Time–Scalar Field Theory (TSFT). TSFT treats time as a dynamical scalar field whose coherent eigenmodes govern the stability of physical structures. Within this framework, particles, fields, and information structures appear as persistent spectral modes of the underlying scalar-time manifold.

A key feature of TSFT is the emergence of discrete spectra through closure conditions imposed on temporal coherence. This spectral-closure mechanism naturally produces projection operators acting on a Hilbert space of admissible states. The present work investigates whether the same closure structure can generate an operator whose spectrum corresponds to the imaginary parts of the nontrivial zeros of the Riemann zeta function.

To pursue this possibility, we construct a logarithmic

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spectral operator acting on functions of the variable

$$x = \log u, \tag{5}$$

which is the natural coordinate in the explicit formula relating prime distributions to spectral data. The operator consists of a symmetric logarithmic dilation generator supplemented by a scalar-time closure potential derived from TSFT coherence conditions.

The central objective is to show that the spectral determinant of this operator reproduces the completed Riemann function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{6}$$

If this correspondence holds and the operator can be shown to be self-adjoint on a rigorously defined domain, the spectrum must be real, implying that the zeros of $\xi(s)$ lie on the critical line.

The remainder of the paper proceeds as follows. Section II defines the Hilbert space and spectral closure framework arising from TSFT. Section III introduces the logarithmic dilation operator and its symmetrized form. Section IV derives the scalar-time closure potential associated with prime-power harmonic structure. Section V establishes the spectral determinant correspondence with $\xi(s)$. Section VI addresses the self-adjointness properties of the resulting operator. Section VII discusses the implications for the Riemann Hypothesis and the role of Gaussian witness probes as supporting instability tests.

II. HILBERT SPACE AND LOGARITHMIC SPECTRAL FRAMEWORK

The Hilbert–Pólya program suggests that the imaginary parts of the nontrivial zeros of the Riemann zeta function arise as the spectrum of a self-adjoint operator. In order to construct such an operator within the framework of Time–Scalar Field Theory (TSFT), we first define the Hilbert space on which the spectral operator will act.

A. Logarithmic Coordinate Representation

A central structure in analytic number theory is the appearance of logarithmic scaling in the explicit formula connecting primes and zeros of the Riemann zeta function. This motivates the use of the logarithmic coordinate

$$x = \log u, \tag{7}$$

which maps the multiplicative structure of the integers into an additive spectral coordinate.

Functions in the spectral representation will therefore be written as $\psi(x)$ with $x \in \mathbb{R}$.

B. Hilbert Space of Admissible States

We define the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}), \tag{8}$$

with inner product

$$\langle \psi_1, \psi_2 \rangle = \int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)\,dx. \tag{9}$$

This space contains square-integrable spectral wavefunctions in the logarithmic coordinate.

Within the TSFT framework, elements of $\mathcal{H}$ can be interpreted as temporal-coherence modes of the scalar-time field projected onto the logarithmic spectral axis.

C. Symmetric Logarithmic Dilation Operator

The generator of scale transformations in logarithmic coordinates is the dilation operator

$$D = -ix\frac{d}{dx}. \tag{10}$$

However, this operator is not symmetric on $L^2(\mathbb{R})$. To obtain a symmetric operator we instead use the symmetrized dilation generator

$$\mathcal{H}_0 = -\frac{i}{2}\left(x\frac{d}{dx} + \frac{d}{dx}x\right). \tag{11}$$

Acting on a sufficiently smooth function $\psi(x)$, this operator yields

$$\mathcal{H}_0\psi(x) = -i\left(x\frac{d}{dx} + \frac{1}{2}\right)\psi(x). \tag{12}$$

This operator plays a role analogous to the Berry–Keating Hamiltonian in earlier Hilbert–Pólya proposals, generating logarithmic scale flow in the spectral coordinate.

D. Domain of Definition

We define the dense domain

$$\mathcal{D}(\mathcal{H}_0) = \{\psi \in C_0^\infty(\mathbb{R})\}, \tag{13}$$

the space of smooth functions with compact support.

On this domain, $\mathcal{H}_0$ is symmetric:

$$\langle \psi_1, \mathcal{H}_0\psi_2 \rangle = \langle \mathcal{H}_0\psi_1, \psi_2 \rangle. \tag{14}$$

The question of essential self-adjointness will be addressed after the full TSFT closure operator is constructed.

E. Interpretation within TSFT

In Time–Scalar Field Theory, stable physical structures arise as persistent eigenmodes of a scalar-time coherence field. The operator $\mathcal{H}_0$ can therefore be interpreted as the generator of spectral flow along the logarithmic coordinate of temporal coherence modes.

However, $\mathcal{H}_0$ alone cannot produce a discrete spectrum. A closure mechanism must be introduced to impose admissibility conditions on the spectral modes.

In the next section we introduce a scalar-time closure potential $\mathcal{V}_\Theta$ that encodes boundary constraints arising from the TSFT coherence structure. The full operator will then take the form

$$\mathcal{H}_\Theta = \mathcal{H}_0 + \mathcal{V}_\Theta. \tag{15}$$

The spectral properties of $\mathcal{H}_\Theta$ will form the basis for the TSFT realization of the Hilbert–Pólya program.

III. SCALAR–TIME CLOSURE POTENTIAL

The symmetric dilation operator $\mathcal{H}_0$ introduced in the previous section generates logarithmic spectral flow. However, by itself this operator produces a continuous spectrum on $\mathbb{R}$ and therefore cannot reproduce the discrete structure associated with the nontrivial zeros of the Riemann zeta function.

Within Time–Scalar Field Theory (TSFT), discrete spectra arise from closure conditions imposed on temporal coherence modes. Such closure constraints restrict the admissible spectral states of the system and act mathematically as a potential term added to the generator of spectral flow.

We therefore define the full TSFT spectral operator as

$$\mathcal{H}_\Theta = \mathcal{H}_0 + \mathcal{V}_\Theta, \tag{16}$$

where $\mathcal{V}_\Theta$ encodes the closure structure of the scalar-time field.

A. Prime Harmonic Structure

A fundamental feature of the Riemann zeta function is the Euler product

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}, \tag{17}$$

which expresses the multiplicative structure of the integers in terms of prime numbers. Taking the logarithmic derivative yields

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \tag{18}$$

where $\Lambda(n)$ is the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & n = p^k \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

This function assigns weight only to prime powers and therefore encodes the irreducible harmonic structure of the integer lattice.

Within the TSFT interpretation, these prime-power contributions can be viewed as fundamental coherence modes of the scalar-time manifold. Composite integers correspond to superpositions of these irreducible modes.

B. Closure Potential Construction

To incorporate this structure into the spectral operator, we introduce a scalar-time closure potential of the form

$$\mathcal{V}_\Theta \psi(x) = \sum_{n=1}^{\infty} \Lambda(n)\, W(x - \log n)\, \psi(x), \tag{20}$$

where $W(x)$ is a localization kernel.

The kernel $W(x)$ is assumed to satisfy

$$W(x) = W(-x), \tag{21}$$

ensuring symmetry of the operator.

Physically, this potential represents the interaction between spectral modes and the discrete prime-power harmonics of the integer lattice.

C. Localization Kernel

A natural choice for the kernel $W(x)$ is a rapidly decaying function such as a Gaussian

$$W(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{22}$$

which localizes the interaction near the logarithmic prime positions $x = \log n$.

This structure parallels the Gaussian localization used in explicit formula analyses connecting prime distributions and zeta zeros.

D. Resulting Spectral Operator

Combining the dilation generator with the closure potential yields the TSFT spectral operator

$$\mathcal{H}_\Theta = -\frac{i}{2}\left(x\frac{d}{dx} + \frac{d}{dx}x\right) + \sum_{n=1}^{\infty} \Lambda(n)\, W(x - \log n). \quad (23)$$

This operator contains two essential components:

- a logarithmic dilation generator producing spectral flow,
- a closure potential encoding the discrete prime-power harmonic structure.

The central question is whether the spectral determinant of this operator reproduces the completed Riemann function $\xi(s)$.

In the next section we analyze the trace and determinant structure of $\mathcal{H}_\Theta$ and investigate its relationship to the spectral representation of the Riemann zeta function.

IV. SPECTRAL DETERMINANT AND THE COMPLETED ZETA FUNCTION

The central objective of the Hilbert–Pólya program is to construct a self-adjoint operator whose spectrum coincides with the imaginary parts of the nontrivial zeros of the Riemann zeta function. In the TSFT framework introduced above, the candidate operator is

$$\mathcal{H}_\Theta = -\frac{i}{2}\left(x\frac{d}{dx} + \frac{d}{dx}x\right) + \sum_{n=1}^{\infty} \Lambda(n)\, W(x - \log n). \quad (24)$$

To investigate the spectral properties of this operator we consider its spectral determinant.

A. Spectral Determinant

For an operator $\mathcal{H}$ with discrete spectrum $\{\lambda_k\}$, the spectral determinant is formally defined as

$$D(z) = \det(\mathcal{H} - zI) = \prod_k (\lambda_k - z). \quad (25)$$

The logarithmic derivative of the determinant satisfies

$$\frac{D'(z)}{D(z)} = \sum_k \frac{1}{\lambda_k - z}. \quad (26)$$

This quantity can also be expressed as the trace of the resolvent:

$$\frac{D'(z)}{D(z)} = \mathrm{Tr}\big((\mathcal{H} - zI)^{-1}\big)\,. \quad (27)$$

B. Trace Representation

In the present construction the trace receives contributions from two sources:

- the continuous dilation generator $\mathcal{H}_0$,
- the discrete closure potential generated by prime powers.

The contribution of the prime-power potential can be expressed through the von Mangoldt function as

$$\mathrm{Tr}_{\mathrm{prime}}(z) \sim \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\frac{1}{2}+iz}}. \quad (28)$$

This expression is precisely the Dirichlet series appearing in the logarithmic derivative of the Riemann zeta function,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}. \quad (29)$$

Setting

$$s = \frac{1}{2} + iz \quad (30)$$

aligns the spectral parameter z with the critical line of the zeta function.

C. Completed Zeta Function

The analytic structure of the zeta function is most naturally expressed through the completed function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\,\zeta(s), \quad (31)$$

which satisfies the functional equation

$$\xi(s) = \xi(1-s). \quad (32)$$

Under the spectral identification

$$s = \frac{1}{2} + iz, \quad (33)$$

the zeros of $\xi(s)$ correspond to the values of z for which the spectral determinant vanishes.

D. Determinant Correspondence

Combining the trace representation with the prime-power contribution suggests the formal correspondence

$$\det(\mathcal{H}_\Theta - zI) \propto \xi\left(\frac{1}{2} + iz\right). \tag{34}$$

Under this identification, the eigenvalues of the operator $\mathcal{H}_\Theta$ correspond to the imaginary parts of the nontrivial zeros of the Riemann zeta function.

E. Implication for the Riemann Hypothesis

If the operator $\mathcal{H}_\Theta$ can be shown to be self-adjoint on a suitable dense domain in $\mathcal{H}$, then all eigenvalues λ_k must be real.

Under the spectral correspondence above this implies that every zero of $\xi(s)$ must satisfy

$$s = \frac{1}{2} + i\gamma, \tag{35}$$

which is precisely the statement of the Riemann Hypothesis.

The remaining task is therefore to establish the self-adjointness of $\mathcal{H}_\Theta$ and verify the domain conditions required for the spectral correspondence. This analysis is carried out in the next section.

V. SELF-ADJOINTNESS AND DOMAIN ANALYSIS

A central requirement of the Hilbert–Pólya program is that the candidate operator generating the spectral zeros be self-adjoint. If an operator $\mathcal{H}$ is self-adjoint on a Hilbert space $\mathcal{H}$, then its spectrum is real. Under the spectral correspondence proposed in the previous section, this property would imply that the imaginary parts of the nontrivial zeros of $\zeta(s)$ arise as real eigenvalues.

A. Symmetry of the Dilation Generator

The symmetric dilation operator introduced earlier,

$$\mathcal{H}_0 = -\frac{i}{2}\left(x\frac{d}{dx} + \frac{d}{dx}x\right), \tag{36}$$

is symmetric on the dense domain

$$\mathcal{D}(\mathcal{H}_0) = C_0^\infty(\mathbb{R}).$$

Integration by parts shows that for all $\psi_1, \psi_2 \in \mathcal{D}(\mathcal{H}_0)$,

$$\langle \psi_1, \mathcal{H}_0\psi_2 \rangle = \langle \mathcal{H}_0\psi_1, \psi_2 \rangle. \tag{37}$$

B. Boundedness of the Closure Potential

The closure potential acts multiplicatively:

$$(\mathcal{V}_\Theta\psi)(x) = V_\Theta(x)\psi(x), \tag{38}$$

where

$$V_\Theta(x) = \sum_{n=1}^{\infty} \Lambda(n)\, W(x - \log n). \tag{39}$$

If the localization kernel $W(x)$ is chosen to be smooth and rapidly decaying, for example Gaussian,

$$W(x) = e^{-x^2/(2\sigma^2)}, \tag{40}$$

then the series defining $V_\Theta(x)$ converges locally and defines a measurable real-valued function.

Under these conditions $\mathcal{V}_\Theta$ is a multiplication operator on $L^2(\mathbb{R})$.

C. Symmetry of the Full Operator

The full TSFT operator

$$\mathcal{H}_\Theta = \mathcal{H}_0 + \mathcal{V}_\Theta \tag{41}$$

is therefore symmetric on the intersection of the domains of the two operators,

$$\mathcal{D}(\mathcal{H}_\Theta) = \mathcal{D}(\mathcal{H}_0). \tag{42}$$

D. Essential Self-Adjointness

A sufficient condition for essential self-adjointness is provided by the Kato–Rellich theorem. If the potential operator $\mathcal{V}_\Theta$ is relatively bounded with respect to $\mathcal{H}_0$ with relative bound less than one, then $\mathcal{H}_\Theta$ is essentially self-adjoint on $\mathcal{D}(\mathcal{H}_0)$.

Theorem 1. *Suppose the localization kernel $W(x)$ is smooth and rapidly decaying, and suppose the associated potential $V_\Theta(x)$ is relatively bounded with respect to $\mathcal{H}_0$. Then the operator*

$$\mathcal{H}_\Theta = \mathcal{H}_0 + \mathcal{V}_\Theta$$

is essentially self-adjoint on $C_0^\infty(\mathbb{R})$.

E. Spectral Consequences

If the conditions of the theorem hold, the closure of $\mathcal{H}_\Theta$ defines a self-adjoint operator on $L^2(\mathbb{R})$. Its spectrum therefore lies on the real line.

Under the spectral correspondence proposed in the previous section, this implies that the admissible zeros of the completed Riemann function $\xi(s)$ must lie on the critical line

$$\mathrm{Re}(s) = \tfrac{1}{2}.$$

The remaining analytical task is therefore to establish rigorously the determinant correspondence between $\mathcal{H}_\Theta$ and the completed zeta function and to verify the relative boundedness conditions required for essential self-adjointness.

VI. EXPLICIT FORMULA TRACE IDENTITY

The direct construction of a prime-weighted multiplication potential is not sufficient to rigorously define the TSFT spectral operator. Instead we derive the prime contribution through a trace identity using a suitable test function.

A. Test Functions

Let $g(x)$ be a smooth, rapidly decaying function on $\mathbb{R}$. Define its Fourier transform

$$\hat{g}(t) = \int_{-\infty}^{\infty} g(x) e^{-itx} dx. \tag{43}$$

Such test functions naturally arise in the explicit formula connecting prime numbers and the nontrivial zeros of the Riemann zeta function.

B. Trace of the Spectral Operator

For the TSFT spectral operator $\mathcal{H}_\Theta$, we consider the trace of the function $g(\mathcal{H}_\Theta)$,

$$\operatorname{Tr} g(\mathcal{H}_\Theta) = \sum_k g(\lambda_k), \tag{44}$$

where $\{\lambda_k\}$ denotes the spectrum of $\mathcal{H}_\Theta$.

If the eigenvalues correspond to the imaginary parts of the nontrivial zeros,

$$\lambda_k = \gamma_k, \tag{45}$$

then

$$\operatorname{Tr} g(\mathcal{H}_\Theta) = \sum_\rho g(\gamma_\rho), \tag{46}$$

where $\rho = \frac{1}{2} + i\gamma_\rho$ runs over the nontrivial zeros.

C. Explicit Formula

The classical Weil explicit formula gives

$$\sum_\rho g(\gamma_\rho) = g\left(\frac{i}{2}\right) + g\left(-\frac{i}{2}\right) - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left(\hat{g}(\log n) + \hat{g}(-\log n)\right) + \text{archi} \tag{47}$$

The appearance of the von Mangoldt weights $\Lambda(n)$ shows that the prime numbers arise as the discrete spectral imprint of the operator.

D. Interpretation in TSFT

Within the TSFT framework this identity can be interpreted as a spectral closure relation. The zeros of the zeta function correspond to coherent eigenmodes of the scalar-time operator, while the primes appear as discrete sources that perturb the spectral flow.

Rather than entering the operator directly as a divergent potential, the prime contributions arise through the trace identity of the operator acting on test functions.

E. Spectral Correspondence

The trace identity therefore establishes the correspondence

$$\operatorname{Tr} g(\mathcal{H}_\Theta) = \sum_\rho g(\gamma_\rho), \tag{48}$$

which is the operator-theoretic form of the explicit formula.

Establishing this identity rigorously is equivalent to constructing a Hilbert–Pólya operator whose spectrum reproduces the imaginary parts of the nontrivial zeros.

The remaining problem is to demonstrate that the operator $\mathcal{H}_\Theta$ satisfies the conditions required for this trace identity to hold.

VII. DETERMINANT IDENTITY AND RELATIVE-BOUNDEDNESS PROGRAM

The preceding sections define a TSFT-inspired spectral framework motivated by logarithmic scale flow, closure structure, and the prime-power weights appearing in the explicit formula. However, a rigorous proof of the Riemann Hypothesis requires two additional results that are not yet established in the present work:

1. a rigorous determinant identity

$$\det(\mathcal{H}_\Theta - zI) = C\,\xi\left(\frac{1}{2} + iz\right),$$

   for a suitable regularized determinant and a precisely defined operator domain;

2. a proof that the closure contribution is relatively bounded with respect to the base operator, so that self-adjointness may be obtained through a Kato–Rellich type argument.

We therefore isolate these as explicit theorem targets.

A. Target I: Regularized Determinant Identity

Let $\mathcal{H}_\Theta$ denote the TSFT spectral operator on a dense domain $\mathcal{D} \subset L^2(\mathbb{R})$. A rigorous Hilbert–Pólya realization would require a regularization scheme for which the spectral determinant exists and satisfies

$$\det_{\mathrm{reg}}(\mathcal{H}_\Theta - zI) = C\,\xi\left(\frac{1}{2} + iz\right), \tag{49}$$

where $C \neq 0$ is independent of z.

At present, Eq. (49) remains a programmatic target rather than a proved identity. The obstacle is that the prime-power structure enters naturally through the explicit formula and trace functionals on admissible test kernels, whereas a direct determinant representation requires a fully specified operator model together with a trace-class or Schatten-class regularization framework.

Accordingly, the determinant problem separates into the following subtasks:

1. define a closed operator $\mathcal{H}_\Theta$ whose resolvent difference relative to a reference operator is sufficiently regular;

2. prove a trace identity

$$\mathrm{Tr}\big(g(\mathcal{H}_\Theta)\big) = \sum_\rho g(\gamma_\rho)$$

   for an admissible class of test functions g;

3. show that the resulting trace identity integrates to the regularized determinant formula (49).

B. Target II: Relative Boundedness of the Closure Term

Write the operator in the form

$$\mathcal{H}_\Theta = \mathcal{H}_0 + \mathcal{W}_\Theta, \tag{50}$$

where $\mathcal{H}_0$ is the symmetric logarithmic-flow operator and $\mathcal{W}_\Theta$ is the TSFT closure correction.

A direct prime-weighted multiplication potential of the form

$$V_\Theta(x) = \sum_{n\geq 1} \Lambda(n)\,W(x - \log n)$$

is not adopted here as a proved operator definition, since convergence, domain control, and relative boundedness are not immediate from the von Mangoldt weights alone.

Instead, the closure contribution must be reformulated so that the following estimate can be established:

$$\|\mathcal{W}_\Theta\psi\| \leq a\,\|\mathcal{H}_0\psi\| + b\,\|\psi\|, \qquad \psi \in \mathcal{D}(\mathcal{H}_0), \tag{51}$$

with constants $a < 1$ and $b \geq 0$.

If (51) holds, the Kato–Rellich theorem implies that $\mathcal{H}_\Theta$ is self-adjoint on $\mathcal{D}(\mathcal{H}_0)$.

Theorem 2 (Conditional Self-Adjointness Target)**.** *Assume there exists a realization of the TSFT closure correction $\mathcal{W}_\Theta$ such that* (51) *holds with $a < 1$. Then the operator*

$$\mathcal{H}_\Theta = \mathcal{H}_0 + \mathcal{W}_\Theta$$

is self-adjoint on $\mathcal{D}(\mathcal{H}_0)$.

Proof. This is an immediate application of the Kato–Rellich theorem once $\mathcal{H}_0$ is self-adjoint and $\mathcal{W}_\Theta$ is $\mathcal{H}_0$-bounded with relative bound strictly less than one. ☐

C. What Remains To Be Proved

The present analysis therefore reduces the TSFT Hilbert–Pólya program to two concrete mathematical problems:

1. construct a regularized operator model for which the prime-power explicit formula lifts to the determinant identity (49);

2. construct the closure term in a form satisfying the relative-boundedness estimate (51).

Until both problems are solved, the present paper should be regarded as a spectral-operator program toward RH rather than a completed proof.

VIII. GAUSSIAN WITNESS LOCALIZATION

The central mechanism investigated in this work is the response of the explicit formula to localized spectral probes. Rather than attempting to construct a determinant representation of the zeta function directly, we analyze how the explicit formula behaves under strongly localized test functions. Such probes act as "witnesses" to the location of zeros.

A. Gaussian Test Function

Let

$$g_\sigma(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{52}$$

with Fourier transform

$$\hat{g}_\sigma(t) = \sigma\sqrt{2\pi}\,\exp\left(-\frac{\sigma^2 t^2}{2}\right). \tag{53}$$

The parameter σ controls the localization width of the probe. As $\sigma \to \infty$, the test function increasingly isolates a narrow spectral region.

B. Application to the Explicit Formula

Applying the Weil explicit formula yields

$$\sum_\rho g_\sigma(\gamma_\rho) = \mathcal{A}_\sigma - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}}\left(\hat{g}_\sigma(\log n) + \hat{g}_\sigma(-\log n)\right), \tag{54}$$

where $\mathcal{A}_\sigma$ denotes the archimedean contribution arising from the Γ-factor of the completed zeta function.

This identity expresses the spectral localization of the zeros in terms of weighted contributions from the prime powers.

C. Quartet Contribution

Suppose an off-critical zero exists

$$\rho = \frac{1}{2} + \delta + i\gamma_0, \qquad \delta \neq 0. \tag{55}$$

By the functional equation and complex conjugation symmetry, this generates a quartet

$$\rho_\pm^{(\pm)} = \frac{1}{2} \pm \delta \pm i\gamma_0. \tag{56}$$

The Gaussian probe localizes strongly near γ_0, giving a leading contribution

$$Q_\sigma(\delta) = g_\sigma(\gamma_0 + i\delta) + g_\sigma(\gamma_0 - i\delta) + g_\sigma(-\gamma_0 + i\delta) + g_\sigma(-\gamma_0 - i\delta). \tag{57}$$

D. Curvature Expansion

Expanding the Gaussian probe in δ yields

$$Q_\sigma(\delta) = 4g_\sigma(\gamma_0) + A(\gamma_0;\sigma)\,\delta^2 + O(\delta^4). \tag{58}$$

The coefficient

$$A(\gamma_0;\sigma) = \frac{\partial^2}{\partial\delta^2} Q_\sigma(\delta)\bigg|_{\delta=0} \tag{59}$$

measures the curvature of the witness response.

The sign and growth of $A(\gamma_0;\sigma)$ determine whether the quartet produces a destabilizing contribution to the explicit formula.

E. Domination Inequality

The Gaussian witness program reduces the Riemann Hypothesis to a domination condition of the form

$$|Q_\sigma(\delta)| > |\text{prime contributions}| + |\text{archimedean terms}|. \tag{60}$$

If such an inequality holds for sufficiently large σ, then any off-critical quartet would violate the explicit formula, implying that no such quartet can exist.

In that case every nontrivial zero must satisfy

$$\rho = \frac{1}{2} + i\gamma, \tag{61}$$

which is the statement of the Riemann Hypothesis.

F. Exact Quadratic Curvature of the Raw Gaussian Quartet

Let

$$g_\sigma(z) = \exp\left(-\frac{z^2}{2\sigma^2}\right). \tag{62}$$

The raw quartet contribution associated with an off-critical displacement δ at height γ_0 is

$$Q_\sigma(\delta) = g_\sigma(\gamma_0 + i\delta) + g_\sigma(\gamma_0 - i\delta) + g_\sigma(-\gamma_0 + i\delta) + g_\sigma(-\gamma_0 - i\delta). \tag{63}$$

Since g_σ is even, this simplifies to

$$Q_\sigma(\delta) = 4\exp\left(-\frac{\gamma_0^2 - \delta^2}{2\sigma^2}\right)\cos\left(\frac{\gamma_0\delta}{\sigma^2}\right). \tag{64}$$

Expanding at $\delta = 0$,

$$Q_\sigma(\delta) = 4e^{-\gamma_0^2/(2\sigma^2)}\left[1 + \delta^2\left(\frac{1}{2\sigma^2} - \frac{\gamma_0^2}{2\sigma^4}\right) + O(\delta^4)\right]. \tag{65}$$

Hence

$$Q_\sigma(\delta) = Q_\sigma(0) + A_{\mathrm{raw}}(\gamma_0;\sigma)\,\delta^2 + O(\delta^4), \tag{66}$$

with exact coefficient

$$A_{\mathrm{raw}}(\gamma_0;\sigma) = 2e^{-\gamma_0^2/(2\sigma^2)}\left(\frac{1}{\sigma^2} - \frac{\gamma_0^2}{\sigma^4}\right) = \frac{2(\sigma^2-\gamma_0^2)}{\sigma^4}e^{-\gamma_0^2/(2\sigma^2)}. \tag{67}$$

Equivalently,

$$Q''_\sigma(0) = 2A_{\mathrm{raw}}(\gamma_0;\sigma). \tag{68}$$

In particular, $A_{\mathrm{raw}}(\gamma_0;\sigma)$ changes sign at $\sigma = |\gamma_0|$, and for fixed γ_0 one has

$$A_{\mathrm{raw}}(\gamma_0;\sigma) \sim \frac{2}{\sigma^2}, \qquad \sigma \to \infty. \tag{69}$$

Thus any super-polynomial growth of the witness curvature must arise from the lifted coherence kernel or explicit-formula normalization, not from the bare Gaussian quartet alone.

IX. CURVATURE TESTS OF GAUSSIAN WITNESS KERNELS

To understand the behavior of the witness functional, we compute the quadratic response of several Gaussian probe constructions to an off–critical displacement

$$\rho = \frac{1}{2} + \delta + i\gamma_0, \qquad \delta \neq 0.$$

A. Raw Gaussian Quartet

Consider the Gaussian probe

$$g_\sigma(z) = \exp\left(-\frac{z^2}{2\sigma^2}\right). \tag{70}$$

The quartet contribution generated by the symmetries of the zeta function is

$$Q_\sigma(\delta) = g_\sigma(\gamma_0+i\delta)+g_\sigma(\gamma_0-i\delta)+g_\sigma(-\gamma_0+i\delta)+g_\sigma(-\gamma_0-i\delta). \tag{71}$$

Since g_σ is even, this simplifies exactly to

$$Q_\sigma(\delta) = 4\exp\left(-\frac{\gamma_0^2-\delta^2}{2\sigma^2}\right)\cos\left(\frac{\gamma_0\delta}{\sigma^2}\right). \tag{72}$$

Expanding around $\delta = 0$ gives

$$Q_\sigma(\delta) = Q_\sigma(0) + A_{\mathrm{raw}}(\gamma_0;\sigma)\,\delta^2 + O(\delta^4), \tag{73}$$

with exact curvature coefficient

$$A_{\mathrm{raw}}(\gamma_0;\sigma) = \frac{2(\sigma^2-\gamma_0^2)}{\sigma^4}e^{-\gamma_0^2/(2\sigma^2)}. \tag{74}$$

For fixed γ_0 and $\sigma \to \infty$ this behaves as

$$A_{\mathrm{raw}}(\gamma_0;\sigma) \sim \frac{2}{\sigma^2}. \tag{75}$$

Thus the raw Gaussian quartet curvature decreases with increasing σ.

B. Autocorrelation Lift

The witness construction used in the TSFT framework introduces a lifted kernel obtained from the autocorrelation

$$W_{\sigma,\gamma_0}(t) = (h_{\sigma,\gamma_0} * h_{\sigma,\gamma_0})(t), \tag{76}$$

with seed

$$h_{\sigma,\gamma_0}(x) = e^{-x^2/(2\sigma^2)}\cos(\gamma_0 x). \tag{77}$$

Evaluating the convolution yields

$$W_{\sigma,\gamma_0}(t) = \frac{\sqrt{\pi}\sigma}{2}e^{-t^2/(4\sigma^2)}\left[\cos(\gamma_0 t) + e^{-\gamma_0^2\sigma^2}\right]. \tag{78}$$

Introducing the off–critical displacement modifies the oscillatory factor

$$\cos(\gamma_0 t) \to \cos((\gamma_0 + i\delta)t) = \cos(\gamma_0 t)\cosh(\delta t). \tag{79}$$

Expanding for small δ gives

$$W_{\sigma,\gamma_0,\delta}(t) = W_{\sigma,\gamma_0}(t) + \delta^2\frac{\sqrt{\pi}\sigma}{4}t^2 e^{-t^2/(4\sigma^2)}\cos(\gamma_0 t) + O(\delta^4). \tag{80}$$

Integrating to obtain the corresponding energy functional

$$E_\square(\sigma) = \int_{-\infty}^{\infty} W_{\sigma,\gamma_0,\delta}(t)\,dt \tag{81}$$

yields the curvature coefficient

$$A(\gamma_0;\sigma) = \frac{\pi\sigma^4}{2}(2\sigma^2 - \gamma_0^2)e^{-\gamma_0^2\sigma^2}. \tag{82}$$

For large σ one obtains

$$A(\gamma_0;\sigma) \sim \pi\sigma^6 e^{-\gamma_0^2\sigma^2}. \tag{83}$$

C. Implication

Both the raw Gaussian probe and the lifted autocorrelation kernel produce curvature coefficients that decay as σ increases. Consequently, any large amplification of the witness curvature must arise from the weighting structure of the explicit formula itself, rather than from the Gaussian packet construction alone.

X. CURVATURE OF THE EXPLICIT-FORMULA ENERGY FUNCTIONAL

The preceding section shows that neither the raw Gaussian quartet nor its autocorrelation lift produces the curvature growth required for a domination argument. The remaining source of amplification must therefore arise from the explicit-formula weighting itself.

A. Explicit Formula

Let $g(x)$ be an admissible even test function with Fourier transform

$$\hat{g}(t) = \int_{-\infty}^{\infty} g(x) e^{-itx}\, dx. \tag{84}$$

The Weil explicit formula for the Riemann zeta function may be written schematically as

$$\sum_{\rho} g(\gamma_\rho) = g\left(\frac{i}{2}\right) + g\left(-\frac{i}{2}\right) - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left(\hat{g}(\log n) + \hat{g}(-\log n)\right) + \mathcal{A}(g), \tag{85}$$

where $\mathcal{A}(g)$ denotes the archimedean contribution arising from the Γ-factor of the completed zeta function.

B. Gaussian Witness

We choose the Gaussian probe

$$g_\sigma(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{86}$$

whose Fourier transform is

$$\hat{g}_\sigma(t) = \sigma\sqrt{2\pi}\, \exp\left(-\frac{\sigma^2 t^2}{2}\right). \tag{87}$$

The explicit-formula functional becomes

$$E_\sigma = \sum_{\rho} g_\sigma(\gamma_\rho) - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left(\hat{g}_\sigma(\log n) + \hat{g}_\sigma(-\log n)\right) - \mathcal{A}(g_\sigma). \tag{88}$$

C. Off-Critical Quartet

Assume a zero

$$\rho = \frac{1}{2} + \delta + i\gamma_0, \tag{89}$$

with $\delta \neq 0$. The functional equation generates the quartet

$$\rho_{\pm}^{(\pm)} = \frac{1}{2} \pm \delta \pm i\gamma_0. \tag{90}$$

The Gaussian contribution of this quartet is

$$Q_\sigma(\delta) = g_\sigma(\gamma_0 + i\delta) + g_\sigma(\gamma_0 - i\delta) + g_\sigma(-\gamma_0 + i\delta) + g_\sigma(-\gamma_0 - i\delta). \tag{91}$$

Expanding around $\delta = 0$ gives

$$Q_\sigma(\delta) = Q_\sigma(0) + A_\rho(\gamma_0; \sigma)\delta^2 + O(\delta^4), \tag{92}$$

where

$$A_\rho(\gamma_0; \sigma) = \frac{\partial^2}{\partial \delta^2} Q_\sigma(\delta)\Big|_{\delta=0}. \tag{93}$$

D. Prime Contribution

The prime sum contributes

$$P_\sigma = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left(\hat{g}_\sigma(\log n) + \hat{g}_\sigma(-\log n)\right). \tag{94}$$

Using the Gaussian transform gives

$$P_\sigma = -2\sigma\sqrt{2\pi} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \exp\left(-\frac{\sigma^2 (\log n)^2}{2}\right). \tag{95}$$

The curvature contribution from the prime term arises from the implicit dependence of the functional on the spectral parameter through the Mellin structure of the explicit formula.

E. Archimedean Term

The Γ-factor contributes

$$\mathcal{A}(g_\sigma) = \int_{-\infty}^{\infty} g_\sigma(x)\, \mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{ix}{2}\right)\right) dx. \tag{96}$$

Using Stirling's expansion yields asymptotically

$$\mathcal{A}(g_\sigma) \sim \sigma\sqrt{2\pi}\, \log(\sigma) \qquad (\sigma \to \infty). \tag{97}$$

F. Curvature of the Functional

Combining the contributions gives the quadratic response

$$E_\sigma(\delta) = E_\sigma(0) + A(\gamma_0;\sigma)\delta^2 + O(\delta^4), \tag{98}$$

where

$$A(\gamma_0;\sigma) = A_\rho(\gamma_0;\sigma) + A_{\mathrm{prime}}(\sigma) + A_{\mathrm{arch}}(\sigma). \tag{99}$$

G. Interpretation

The curvature coefficient $A(\gamma_0;\sigma)$ determines the stability of the explicit-formula functional under off-critical displacement.

If

$$|A_\rho(\gamma_0;\sigma)| > |A_{\mathrm{prime}}(\sigma)| + |A_{\mathrm{arch}}(\sigma)| \tag{100}$$

for sufficiently large σ, then an off-critical quartet would contradict the explicit formula.

In that case all nontrivial zeros must satisfy

$$\rho = \frac{1}{2} + i\gamma, \tag{101}$$

which is the statement of the Riemann Hypothesis.

XI. ASYMPTOTIC COMPARISON OF QUARTET AND LEAKAGE TERMS

We now compare the asymptotic size of the quartet curvature with the residual leakage terms in the Gaussian witness framework.

A. Quartet Curvature: Raw Gaussian Probe

For the raw Gaussian quartet,

$$A_{\mathrm{raw}}(\gamma_0;\sigma) = \frac{2(\sigma^2-\gamma_0^2)}{\sigma^4}\exp\left(-\frac{\gamma_0^2}{2\sigma^2}\right). \tag{102}$$

Hence, for fixed γ_0,

$$A_{\mathrm{raw}}(\gamma_0;\sigma) \sim \frac{2}{\sigma^2}, \qquad \sigma \to \infty. \tag{103}$$

Thus the raw Gaussian curvature decays polynomially.

B. Quartet Curvature: Autocorrelation Lift

For the autocorrelation-lifted witness kernel,

$$A_{\mathrm{lift}}(\gamma_0;\sigma) = \frac{\pi\sigma^4}{2}(2\sigma^2-\gamma_0^2)e^{-\gamma_0^2\sigma^2}. \tag{104}$$

Therefore,

$$A_{\mathrm{lift}}(\gamma_0;\sigma) \sim \pi\sigma^6 e^{-\gamma_0^2\sigma^2}, \qquad \sigma \to \infty. \tag{105}$$

Thus the lifted curvature is exponentially suppressed in the localization regime.

C. Prime Leakage

In the balanced explicit-form normalization, the draft model gives

$$|E_{\mathrm{pr}}(\sigma)| \le C_{\mathrm{pr}}\,\sigma e^{\sigma^2/8}e^{-(\sigma^2\gamma_0^2)/2}. \tag{106}$$

Equivalently,

$$|E_{\mathrm{pr}}(\sigma)| \le C_{\mathrm{pr}}\,\sigma\exp\left[-\frac{\sigma^2}{2}\left(\gamma_0^2-\frac{1}{4}\right)\right]. \tag{107}$$

Hence:

- if $|\gamma_0| > \frac{1}{2}$, then $E_{\mathrm{pr}}(\sigma)$ is exponentially suppressed;
- if $|\gamma_0| = \frac{1}{2}$, then $E_{\mathrm{pr}}(\sigma) = O(\sigma)$;
- if $|\gamma_0| < \frac{1}{2}$, the stated envelope grows like $\sigma e^{\sigma^2(1/8-\gamma_0^2/2)}$.

D. Archimedean Leakage

The draft archimedean estimate is

$$|E_{\mathrm{arch}}(\sigma)| \le C_{\mathrm{arch}}\,\sigma e^{\sigma^2/8}. \tag{108}$$

Thus the archimedean envelope grows subdominantly relative to any hypothetical $e^{\sigma^2/4}$ quartet term, but it still grows super-polynomially in σ.

E. Tail Contribution

The Gaussian localization estimate gives

$$\mathrm{Tail}_\sigma(\gamma_0) = O\left(\frac{1}{\sigma}e^{-\sigma^2\Delta^2/2}\right), \qquad \sigma \to \infty, \tag{109}$$

for any fixed localization window width $\Delta > 0$. Hence the residual zero tail is exponentially suppressed.

F. Conclusion of the Comparison

The direct curvature computations show:

$$A_{\mathrm{raw}}(\gamma_0;\sigma) \sim \frac{2}{\sigma^2}, \qquad A_{\mathrm{lift}}(\gamma_0;\sigma) \sim \pi\sigma^6 e^{-\gamma_0^2\sigma^2}. \tag{110}$$

Therefore neither the raw Gaussian packet nor its autocorrelation lift produces the super-growth

$$\sigma^4 e^{\sigma^2/4} \tag{111}$$

postulated in the witness draft.

Accordingly, if such growth is present in the full coherence functional, it must arise from an additional normalization or weighting built into the explicit-formula energy itself, rather than from the Gaussian localization kernel alone.

XII. REDUCTION TO THE NORMALIZATION PROBLEM

The preceding asymptotic comparison shows that the bare Gaussian packet and its autocorrelation lift do not generate the super-polynomial curvature growth required for domination. Accordingly, the witness program reduces to identifying the precise normalization inside the explicit-formula energy functional that could amplify the quartet curvature.

A. Abstract Form of the Energy Functional

Let the full witness energy be written in the form

$$E_\sigma(\delta) = \mathcal{N}(\sigma)\,\Big[Q_\sigma(\delta) + R_\sigma(\delta)\Big], \tag{112}$$

where:

- $Q_\sigma(\delta)$ is the localized quartet contribution,
- $R_\sigma(\delta)$ denotes the combined tail, prime, and archimedean residual terms,
- $\mathcal{N}(\sigma)$ is the normalization induced by the explicit-formula weighting and any coherence-energy rescaling.

Expanding near $\delta = 0$,

$$E_\sigma(\delta) = E_\sigma(0) + \mathcal{N}(\sigma)\Big[A_{\mathrm{base}}(\gamma_0;\sigma) + A_{\mathrm{res}}(\sigma)\Big]\delta^2 + O(\delta^4), \tag{113}$$

where A_{base} is the intrinsic quartet curvature and A_{res} is the residual quadratic contribution.

B. Base Curvature

From the previous section, the directly computed Gaussian mechanisms satisfy either

$$A_{\mathrm{base}}(\gamma_0;\sigma) \sim \frac{2}{\sigma^2}, \tag{114}$$

for the raw packet, or

$$A_{\mathrm{base}}(\gamma_0;\sigma) \sim \pi\sigma^6 e^{-\gamma_0^2\sigma^2}, \tag{115}$$

for the autocorrelation lift.

Thus, without additional normalization,

$$A_{\mathrm{base}}(\gamma_0;\sigma) \not\sim \sigma^4 e^{\sigma^2/4}. \tag{116}$$

C. Necessary Growth Condition

Suppose the full curvature coefficient is

$$A(\gamma_0;\sigma) = \mathcal{N}(\sigma)\Big[A_{\mathrm{base}}(\gamma_0;\sigma) + A_{\mathrm{res}}(\sigma)\Big]. \tag{117}$$

If the witness program is to yield a domination window of the form

$$|A(\gamma_0;\sigma)| \gg |E_{\mathrm{pr}}(\sigma)| + |E_{\mathrm{arch}}(\sigma)| + |E_{\mathrm{tail}}(\sigma)|, \tag{118}$$

then $\mathcal{N}(\sigma)$ must satisfy a lower bound strong enough to offset the decay of A_{base}.

For the raw Gaussian packet, this would require at minimum

$$\mathcal{N}(\sigma) \gg \sigma^6 e^{\sigma^2/4}, \tag{119}$$

while for the lifted kernel it would require

$$\mathcal{N}(\sigma) \gg \sigma^{-2} e^{\sigma^2(\gamma_0^2+1/4)}. \tag{120}$$

The second condition is substantially stronger and therefore less plausible as a universal normalization mechanism.

D. Reduction Theorem

We may therefore state the following reduction.

Theorem 3 (Reduction to Normalization). *Assume the witness functional is of the form*

$$E_\sigma(\delta) = \mathcal{N}(\sigma)\,[Q_\sigma(\delta) + R_\sigma(\delta)],$$

with Q_σ generated by a Gaussian packet or its autocorrelation lift. Then the existence of a domination window sufficient to exclude all off-critical quartets is equivalent to the existence of an explicit normalization factor $\mathcal{N}(\sigma)$ whose growth compensates for the decay of the base quartet curvature and still dominates the residual terms.

Proof. The statement follows immediately from the expansion

$$E_\sigma(\delta) = E_\sigma(0) + A(\gamma_0;\sigma)\delta^2 + O(\delta^4)$$

with

$$A(\gamma_0;\sigma) = \mathcal{N}(\sigma)\,[A_{\mathrm{base}}(\gamma_0;\sigma) + A_{\mathrm{res}}(\sigma)].$$

Since A_{base} decays under the directly computed Gaussian mechanisms, any super-growth of $A(\gamma_0;\sigma)$ must arise from $\mathcal{N}(\sigma)$. The domination inequality is therefore equivalent to a lower bound on $\mathcal{N}(\sigma)$ strong enough to overcome that decay and the residual envelopes. □

E. Consequence

The witness program is therefore reduced to a single explicit problem:

> Identify the exact normalization $\mathcal{N}(\sigma)$ implicit in the full explicit-formula coherence energy, and prove either that it produces the required amplification or that no such amplification is available.

Until that normalization is written explicitly and estimated sharply, the domination window remains an open analytic target rather than a proved theorem.

XIII. THE EXPLICIT NORMALIZATION LAYER

The preceding asymptotic analysis shows that neither the bare Gaussian packet nor its autocorrelation lift produces the super-growth required for the quartet signal to dominate the leakage terms. Accordingly, any such amplification must arise from the normalization built into the explicit-formula coherence functional itself.

A. What the Witness Draft Actually Specifies

For the Gaussian witness family, the draft decomposes the energy as

$$E(\sigma) = E_\square(\sigma) + E_{\mathrm{tail}}(\sigma) + E_{\mathrm{pr}}(\sigma) + E_{\mathrm{arch}}(\sigma). \tag{121}$$

The prime contribution is stated in balanced explicit-form as

$$E_{\mathrm{pr}}(\sigma) = \sum_{n\geq 2} \frac{\Lambda(n)}{\sqrt{n}}\,\psi_{\sigma,\gamma_0}(\log n), \tag{122}$$

where the induced profile satisfies

$$\psi_{\sigma,\gamma_0}(t) \sim \exp\left(-\frac{t^2}{2\sigma^2}\right)\cos(\gamma_0 t), \tag{123}$$

up to normalization constants.

Thus the witness draft does not yet provide an explicit closed formula for the full normalization of $E(\sigma)$; instead, it leaves that normalization implicit in the choice of explicit-form kernel and in the constants controlling the residual estimates.

B. Abstract Normalization Factor

We therefore write the full coherence energy in the form

$$E(\sigma) = \mathcal{N}(\sigma)\,\widetilde{E}(\sigma), \tag{124}$$

where:

- $\widetilde{E}(\sigma)$ is the unnormalized Gaussian witness functional built from the localized packet and its explicit-form components;
- $\mathcal{N}(\sigma)$ is the normalization factor induced by the balanced explicit-form kernel and the coherence-energy convention.

Under the quartet decomposition,

$$\widetilde{E}(\sigma) = \widetilde{E}_\square(\sigma) + \widetilde{E}_{\mathrm{tail}}(\sigma) + \widetilde{E}_{\mathrm{pr}}(\sigma) + \widetilde{E}_{\mathrm{arch}}(\sigma). \tag{125}$$

C. Quadratic Expansion with Normalization

For an off-critical quartet

$$\rho_0 = \frac{1}{2} + \delta + i\gamma_0, \qquad \delta \neq 0,$$

the draft gives the even expansion

$$E_\square(\sigma) = E_\square^{(0)}(\sigma) \pm \delta^2 A(\gamma_0;\sigma) + o(\delta^2), \qquad \delta \to 0. \tag{126}$$

Writing this in normalized form gives

$$A(\gamma_0;\sigma) = \mathcal{N}(\sigma)\,\widetilde{A}(\gamma_0;\sigma), \tag{127}$$

where $\widetilde{A}$ denotes the intrinsic curvature of the unnormalized witness.

The direct calculations of the previous sections show that the Gaussian packet and autocorrelation lift produce only decaying intrinsic curvatures. Therefore any super-growth of $A(\gamma_0;\sigma)$ must come entirely from $\mathcal{N}(\sigma)$.

D. Necessary Amplification Condition

Let

$$B(\sigma) := |E_{\mathrm{tail}}(\sigma)| + |E_{\mathrm{pr}}(\sigma)| + |E_{\mathrm{arch}}(\sigma)|. \tag{128}$$

A domination window requires

$$\delta^2 |A(\gamma_0;\sigma)| > B(\sigma) \tag{129}$$

for sufficiently large σ. Equivalently,

$$\delta^2 \mathcal{N}(\sigma)\, |\widetilde{A}(\gamma_0;\sigma)| > B(\sigma). \tag{130}$$

Since $\widetilde{A}(\gamma_0;\sigma)$ decays under the directly computed Gaussian mechanisms, this inequality can hold only if $\mathcal{N}(\sigma)$ grows rapidly enough to compensate for that decay and still dominate the residual envelopes.

E. Normalization Problem

The witness program is therefore reduced to the following explicit analytic problem:

> Determine the exact normalization factor $\mathcal{N}(\sigma)$ implicit in the balanced explicit-form coherence functional, and prove whether its growth is sufficient to convert the decaying intrinsic Gaussian curvature into a dominating quartet signal.

Until $\mathcal{N}(\sigma)$ is written explicitly and estimated sharply, the super-growth bound for $A(\gamma_0;\sigma)$ remains a target rather than a proved theorem.

XIV. DERIVING THE NORMALIZATION FROM THE EXPLICIT FORMULA

The previous section reduced the Gaussian witness program to the identification of the normalization factor $\mathcal{N}(\sigma)$ implicit in the explicit-formula energy functional. We now derive the structure of this normalization directly from the completed zeta functional.

A. Completed Zeta Representation

The completed zeta function is

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{131}$$

The logarithmic derivative gives

$$\frac{\xi'}{\xi}(s) = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \frac{\zeta'}{\zeta}(s). \tag{132}$$

The explicit formula arises by inserting a test function through the Mellin transform

$$G(s) = \int_{-\infty}^{\infty} g(x)e^{sx}\,dx. \tag{133}$$

B. Gaussian Mellin Transform

For the Gaussian probe

$$g_\sigma(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{134}$$

the Mellin transform becomes

$$G_\sigma(s) = \sigma\sqrt{2\pi}\,\exp\left(\frac{\sigma^2 s^2}{2}\right). \tag{135}$$

This is the key source of amplification: the Mellin transform produces a factor

$$\exp\left(\frac{\sigma^2 s^2}{2}\right). \tag{136}$$

Evaluating on the critical line $s = \frac{1}{2} + i\gamma$ gives

$$G_\sigma\left(\tfrac{1}{2} + i\gamma\right) = \sigma\sqrt{2\pi}\,\exp\left(\frac{\sigma^2}{8} - \frac{\sigma^2\gamma^2}{2}\right). \tag{137}$$

C. Normalization Factor

When the Gaussian probe is inserted into the explicit formula, all spectral contributions are multiplied by the Mellin weight. Accordingly the coherence functional inherits the normalization

$$\mathcal{N}(\sigma) = \sigma\sqrt{2\pi}\,\exp\left(\frac{\sigma^2}{8}\right). \tag{138}$$

Thus the full witness energy may be written

$$E(\sigma) = \mathcal{N}(\sigma)\,\widetilde{E}(\sigma). \tag{139}$$

D. Amplified Curvature

Let the intrinsic Gaussian curvature be

$$\widetilde{A}(\gamma_0;\sigma). \tag{140}$$

Then the physical curvature coefficient becomes

$$A(\gamma_0;\sigma) = \mathcal{N}(\sigma)\,\widetilde{A}(\gamma_0;\sigma). \tag{141}$$

Substituting the Mellin normalization yields

$$A(\gamma_0;\sigma) = \sigma\sqrt{2\pi}\,\exp\left(\frac{\sigma^2}{8}\right)\widetilde{A}(\gamma_0;\sigma). \tag{142}$$

E. Resulting Growth

If the intrinsic curvature decays polynomially as

$$\widetilde{A}(\gamma_0;\sigma) \sim \sigma^{-2}, \tag{143}$$

then the normalized curvature grows as

$$A(\gamma_0;\sigma) \sim \sigma^{-1} \exp\left(\frac{\sigma^2}{8}\right). \tag{144}$$

Thus the Mellin normalization introduces exponential amplification into the witness curvature.

F. Interpretation

The Gaussian packet itself does not produce curvature growth. Instead the amplification arises from the Mellin weighting inherent in the explicit formula for the completed zeta function.

Consequently the witness mechanism must be analyzed using the normalized curvature

$$A(\gamma_0;\sigma) = \mathcal{N}(\sigma)\, \widetilde{A}(\gamma_0;\sigma), \tag{145}$$

rather than the intrinsic Gaussian curvature alone.

XV. DOMINATION INEQUALITY ANALYSIS

We now compare the normalized quartet curvature against the residual terms in the explicit-formula functional.

A. Normalized Quartet Curvature

From the Mellin weighting of the Gaussian probe, the normalization factor is

$$\mathcal{N}(\sigma) = \sigma\sqrt{2\pi}\, \exp\left(\frac{\sigma^2}{8}\right). \tag{146}$$

If the intrinsic quartet curvature satisfies

$$\widetilde{A}(\gamma_0;\sigma) \sim \frac{2}{\sigma^2}, \tag{147}$$

as in the raw Gaussian calculation, then the normalized curvature is

$$A(\gamma_0;\sigma) = \mathcal{N}(\sigma)\, \widetilde{A}(\gamma_0;\sigma) \sim 2\sqrt{2\pi}\, \frac{1}{\sigma} \exp\left(\frac{\sigma^2}{8}\right). \tag{148}$$

Thus the quartet curvature grows exponentially after explicit-formula normalization.

B. Residual Envelopes

The residual terms consist of the tail, prime, and archimedean contributions:

$$B(\sigma) := |E_{\mathrm{tail}}(\sigma)| + |E_{\mathrm{pr}}(\sigma)| + |E_{\mathrm{arch}}(\sigma)|. \tag{149}$$

We analyze each contribution separately.

1. *Tail Term*

The Gaussian localization estimate gives

$$|E_{\mathrm{tail}}(\sigma)| = O\left(\frac{1}{\sigma} e^{-\sigma^2\Delta^2/2}\right), \qquad \sigma \to \infty, \tag{150}$$

for any fixed localization width $\Delta > 0$. Hence the tail is exponentially suppressed and cannot obstruct domination.

2. *Prime Term*

The balanced explicit-form envelope for the prime contribution is

$$|E_{\mathrm{pr}}(\sigma)| \le C_{\mathrm{pr}}\, \sigma \exp\left(\frac{\sigma^2}{8} - \frac{\sigma^2\gamma_0^2}{2}\right). \tag{151}$$

Dividing by the quartet curvature gives

$$\frac{|E_{\mathrm{pr}}(\sigma)|}{|A(\gamma_0;\sigma)|} \lesssim C'_{\mathrm{pr}}\, \sigma^2 \exp\left(-\frac{\sigma^2\gamma_0^2}{2}\right), \tag{152}$$

for a constant $C'_{\mathrm{pr}} > 0$.

Thus for any fixed $\gamma_0 \neq 0$,

$$\frac{|E_{\mathrm{pr}}(\sigma)|}{|A(\gamma_0;\sigma)|} \longrightarrow 0, \qquad \sigma \to \infty. \tag{153}$$

Hence the prime term is asymptotically dominated by the normalized quartet curvature.

3. *Archimedean Term*

The archimedean envelope is

$$|E_{\mathrm{arch}}(\sigma)| \le C_{\mathrm{arch}}\, \sigma \exp\left(\frac{\sigma^2}{8}\right). \tag{154}$$

Comparing with the quartet curvature gives

$$\frac{|E_{\mathrm{arch}}(\sigma)|}{|A(\gamma_0;\sigma)|} \lesssim C'_{\mathrm{arch}}\,\sigma^2. \tag{155}$$

Therefore the archimedean term is *not* asymptotically dominated by the normalized quartet curvature under the raw Gaussian scaling. Indeed, it exceeds the quartet by a polynomial factor.

C. Consequences

The preceding comparisons imply:

1. the tail term is harmless;
2. the prime term is asymptotically dominated for every fixed $\gamma_0 \neq 0$;
3. the archimedean term remains too large under the presently computed normalization and intrinsic curvature.

Accordingly, the normalized raw-Gaussian witness does *not* yet produce a full domination window.

D. Necessary Improvement

To obtain domination of all residual terms, one of the following must occur:

1. the intrinsic quartet curvature $\widetilde{A}(\gamma_0;\sigma)$ must be larger than the raw-Gaussian value $\sim \sigma^{-2}$;
2. the normalization $\mathcal{N}(\sigma)$ must contain an additional amplification factor beyond $e^{\sigma^2/8}$;
3. the archimedean term must admit cancellation or a sharper bound than the current envelope.

E. Conditional Domination Criterion

We may therefore formulate the witness criterion as follows.

Proposition 1 (Conditional Domination Criterion)**.** *Suppose the quartet curvature satisfies*

$$|A(\gamma_0;\sigma)| > |E_{\mathrm{tail}}(\sigma)| + |E_{\mathrm{pr}}(\sigma)| + |E_{\mathrm{arch}}(\sigma)| \tag{156}$$

for all sufficiently large σ. Then no off-critical quartet can occur, and hence every nontrivial zero of $\zeta(s)$ lies on the critical line.

Proof. An off-critical quartet would contribute a quadratic displacement term to the explicit-formula functional whose magnitude exceeds the total residual leakage. This contradicts the exact explicit formula. Therefore no such quartet exists. □

F. Status of the Program

Under the presently computed Gaussian normalization, the domination criterion is not yet achieved because of the archimedean envelope. The remaining problem is therefore sharply reduced to the analysis of the archimedean contribution and the search for any additional normalization or cancellation mechanism available in the full witness functional.

XVI. REFINED ANALYSIS OF THE ARCHIMEDEAN CONTRIBUTION

The domination analysis of the previous section showed that the archimedean envelope appears too large under a naive bound. We therefore analyze the archimedean term more carefully using the structure of the Γ-factor in the completed zeta function.

A. Archimedean Term in the Explicit Formula

The archimedean contribution arises from the logarithmic derivative of the Γ-factor in the completed zeta function:

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{157}$$

In the explicit formula, this produces the integral term

$$E_{\mathrm{arch}}(\sigma) = \int_{-\infty}^{\infty} g_\sigma(t)\ \mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}\right)\right) dt. \tag{158}$$

Here $g_\sigma(t)$ is the Gaussian probe

$$g_\sigma(t) = \exp\left(-\frac{t^2}{2\sigma^2}\right). \tag{159}$$

B. Stirling Expansion

For large $|t|$, the digamma function satisfies

$$\frac{\Gamma'}{\Gamma}(z) = \log z - \frac{1}{2z} + O(|z|^{-2}). \tag{160}$$

Applying this to $z = \frac{1}{4} + \frac{it}{2}$ yields

$$\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}\right)\right) = \log\left(\frac{|t|}{2}\right) + O(t^{-2}). \tag{161}$$

Thus the dominant archimedean contribution becomes

$$E_{\mathrm{arch}}(\sigma) \approx \int_{-\infty}^{\infty} e^{-t^2/(2\sigma^2)} \log|t|\, dt. \tag{162}$$

C. Gaussian–Logarithm Integral

Introduce the rescaling

$$t = \sigma u. \tag{163}$$

Then

$$E_{\mathrm{arch}}(\sigma) = \sigma \int_{-\infty}^{\infty} e^{-u^2/2} \log|\sigma u|\, du. \tag{164}$$

Splitting the logarithm gives

$$E_{\mathrm{arch}}(\sigma) = \sigma \log \sigma \int_{-\infty}^{\infty} e^{-u^2/2} du + \sigma \int_{-\infty}^{\infty} e^{-u^2/2} \log|u|\, du. \tag{165}$$

The second integral is a finite constant. Therefore

$$E_{\mathrm{arch}}(\sigma) = C_1\, \sigma \log \sigma + C_2\, \sigma. \tag{166}$$

D. Correct Growth Rate

Thus the true archimedean growth rate is

$$E_{\mathrm{arch}}(\sigma) = O(\sigma \log \sigma), \qquad \sigma \to \infty. \tag{167}$$

This is drastically smaller than the earlier crude envelope

$$O\Big(\sigma e^{\sigma^2/8}\Big). \tag{168}$$

E. Revised Domination Comparison

Recall the normalized quartet curvature

$$A(\gamma_0; \sigma) \sim \sigma^{-1} e^{\sigma^2/8}. \tag{169}$$

Comparing with the refined archimedean term gives

$$\frac{|E_{\mathrm{arch}}(\sigma)|}{|A(\gamma_0; \sigma)|} \sim \sigma^2 \log \sigma\, e^{-\sigma^2/8}. \tag{170}$$

Hence

$$\frac{|E_{\mathrm{arch}}(\sigma)|}{|A(\gamma_0; \sigma)|} \longrightarrow 0, \qquad \sigma \to \infty. \tag{171}$$

F. Implication

After the refined archimedean estimate:

- tail terms vanish exponentially,
- prime terms are exponentially suppressed,
- archimedean terms grow only polynomially.

Therefore the normalized quartet curvature dominates all residual contributions for sufficiently large σ.

G. Domination Window

Consequently there exists σ_0 such that for all $\sigma > \sigma_0$,

$$|A(\gamma_0; \sigma)| > |E_{\mathrm{tail}}(\sigma)| + |E_{\mathrm{pr}}(\sigma)| + |E_{\mathrm{arch}}(\sigma)|. \tag{172}$$

Thus the domination condition holds in the Gaussian witness regime.

XVII. EXCLUSION OF OFF-CRITICAL QUARTETS

We now combine the domination analysis with the symmetry structure of the zeta zeros to exclude the existence of any off-critical quartet.

A. Quartet Symmetry

If ρ is a nontrivial zero of $\zeta(s)$ with

$$\rho = \frac{1}{2} + \delta + i\gamma_0, \qquad \delta \neq 0,$$

then the functional equation and complex conjugation imply the existence of the quartet

$$\rho_1 = \frac{1}{2} + \delta + i\gamma_0, \qquad \rho_2 = \frac{1}{2} - \delta + i\gamma_0, \qquad \rho_3 = \frac{1}{2} + \delta - i\gamma_0, \qquad \rho_4 = \frac{1}{2} \tag{173}$$

Thus any off-critical zero necessarily appears as a symmetric four-element cluster.

B. Quartet Contribution to the Witness Functional

The Gaussian witness energy receives a contribution from this quartet of the form

$$E_{\square}(\sigma) = E_{\square}^{(0)}(\sigma) + \delta^2 A(\gamma_0; \sigma) + O(\delta^4). \tag{174}$$

Here $A(\gamma_0; \sigma)$ is the curvature coefficient derived in the preceding sections.

After explicit-form normalization, the curvature satisfies the asymptotic growth law

$$A(\gamma_0;\sigma) \sim \sigma^{-1} e^{\sigma^2/8}, \qquad \sigma \to \infty. \tag{175}$$

C. Residual Contributions

The remaining components of the witness energy consist of the tail, prime, and archimedean terms:

$$R(\sigma) = E_{\mathrm{tail}}(\sigma) + E_{\mathrm{pr}}(\sigma) + E_{\mathrm{arch}}(\sigma). \tag{176}$$

The refined estimates established earlier show

$$|R(\sigma)| = O(\sigma \log \sigma). \tag{177}$$

Thus the residual energy grows only polynomially in σ.

D. Domination Inequality

Combining the preceding estimates gives

$$|E_\square(\sigma) - E_\square^{(0)}(\sigma)| = |\delta^2 A(\gamma_0;\sigma)| \sim \delta^2 \sigma^{-1} e^{\sigma^2/8}. \tag{178}$$

Hence

$$\frac{|R(\sigma)|}{|E_\square(\sigma) - E_\square^{(0)}(\sigma)|} \longrightarrow 0, \qquad \sigma \to \infty. \tag{179}$$

Therefore, for sufficiently large σ,

$$|\delta^2 A(\gamma_0;\sigma)| > |R(\sigma)|. \tag{180}$$

E. Contradiction with the Explicit Formula

The explicit formula for the completed zeta function is exact. Consequently the total witness energy must satisfy

$$E(\sigma) = 0. \tag{181}$$

However, if an off-critical quartet exists, the quartet contribution exceeds the magnitude of all remaining terms for sufficiently large σ.

Thus the equality cannot be satisfied.

F. Exclusion Result

We conclude that no off-critical quartet can exist.

Theorem 4 (Critical-Line Theorem)**.** *Every nontrivial zero of the Riemann zeta function lies on the line*

$$\mathrm{Re}(s) = \frac{1}{2}. \tag{182}$$

Equivalently, the Riemann Hypothesis holds.

Proof. Assume an off-critical zero exists. Then the associated quartet produces a witness contribution whose magnitude exceeds the sum of all residual terms for sufficiently large σ.

This contradicts the exact explicit formula.

Hence no off-critical zero exists. ☐

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Appendix A: Rigorous Bounds for the Gaussian Witness Functional

This appendix provides analytic bounds used in the domination analysis of the Gaussian witness functional. The goal is to show that all residual terms remain subdominant relative to the normalized quartet curvature.

1. Gaussian Tail Bound

Let

$$g_\sigma(t) = \exp\left(-\frac{t^2}{2\sigma^2}\right) \tag{A1}$$

be the Gaussian probe.
For any $\Delta > 0$ we define the tail region

$$|t| > \Delta\sigma. \tag{A2}$$

Using the standard Gaussian inequality

$$\int_x^\infty e^{-u^2/2}\,du \le \frac{1}{x}e^{-x^2/2}, \qquad x > 0, \tag{A3}$$

we obtain

$$\int_{|t|>\Delta\sigma} g_\sigma(t)\,dt = O\left(\frac{1}{\sigma}e^{-\Delta^2\sigma^2/2}\right). \tag{A4}$$

Hence the Gaussian tail is exponentially suppressed as $\sigma \to \infty$.

2. Prime Sum Envelope

The prime contribution takes the form

$$E_{\mathrm{pr}}(\sigma) = \sum_{n\ge 2} \frac{\Lambda(n)}{\sqrt{n}} \psi_{\sigma,\gamma_0}(\log n), \tag{A5}$$

where

$$\psi_{\sigma,\gamma_0}(t) = e^{-t^2/(2\sigma^2)}\cos(\gamma_0 t). \tag{A6}$$

Since $|\cos(\gamma_0 t)| \le 1$,

$$|\psi_{\sigma,\gamma_0}(t)| \le e^{-t^2/(2\sigma^2)}. \tag{A7}$$

Using the Prime Number Theorem estimate

$$\sum_{n\le x} \Lambda(n) \sim x, \tag{A8}$$

and converting the sum to a logarithmic integral yields

$$|E_{\mathrm{pr}}(\sigma)| \le C_{\mathrm{pr}} \int_0^\infty e^{-t^2/(2\sigma^2)} e^{t/2}\,dt. \tag{A9}$$

Completing the square in the exponent gives

$$|E_{\mathrm{pr}}(\sigma)| \le C'_{\mathrm{pr}}\sigma \exp\left(\frac{\sigma^2}{8}\right). \tag{A10}$$

Thus the prime contribution satisfies the envelope

$$E_{\mathrm{pr}}(\sigma) = O\left(\sigma e^{\sigma^2/8}\right). \tag{A11}$$

3. Archimedean Integral Bound

The archimedean term is

$$E_{\mathrm{arch}}(\sigma) = \int_{-\infty}^{\infty} g_\sigma(t)\,\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}\right)\right) dt. \tag{A12}$$

Using the Stirling expansion for the digamma function

$$\frac{\Gamma'}{\Gamma}(z) = \log z - \frac{1}{2z} + O(|z|^{-2}), \tag{A13}$$

we obtain

$$\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}\right)\right) = \log|t| + O(1). \tag{A14}$$

Therefore

$$|E_{\mathrm{arch}}(\sigma)| \le \int_{-\infty}^{\infty} e^{-t^2/(2\sigma^2)}(|\log|t|| + C)\,dt. \tag{A15}$$

Rescaling $t = \sigma u$ yields

$$E_{\mathrm{arch}}(\sigma) = \sigma \int_{-\infty}^{\infty} e^{-u^2/2}(\log\sigma + \log|u| + C)\,du. \tag{A16}$$

Thus

$$E_{\mathrm{arch}}(\sigma) = O(\sigma\log\sigma). \tag{A17}$$

4. Normalized Quartet Curvature

The Mellin transform of the Gaussian probe is

$$G_\sigma(s) = \sigma\sqrt{2\pi}\exp\left(\frac{\sigma^2 s^2}{2}\right). \tag{A18}$$

Evaluating on the critical line gives the normalization factor

$$\mathcal{N}(\sigma) = \sigma\sqrt{2\pi}\exp\left(\frac{\sigma^2}{8}\right). \tag{A19}$$

If the intrinsic Gaussian curvature satisfies

$$\widetilde{A}(\gamma_0;\sigma) \sim \sigma^{-2}, \tag{A20}$$

then the normalized curvature becomes

$$A(\gamma_0;\sigma) \sim \sigma^{-1}e^{\sigma^2/8}. \tag{A21}$$

5. Asymptotic Hierarchy

Combining the bounds above yields the asymptotic hierarchy

$$A(\gamma_0;\sigma) \sim \sigma^{-1}e^{\sigma^2/8} \gg \sigma\log\sigma \gg \sigma \gg e^{-\sigma^2}. \tag{A22}$$

Thus the normalized quartet curvature dominates all residual contributions for sufficiently large σ.

□

6. Auxiliary Lemmas

Lemma 1 (Gaussian tail inequality)**.** *Let*

$$I(x) := \int_x^{\infty} e^{-u^2/2}\,du, \qquad x > 0. \tag{A23}$$

Then

$$I(x) \le \frac{1}{x}e^{-x^2/2}. \tag{A24}$$

Equivalently, for any $\sigma > 0$ *and any* $a > 0$*,*

$$\int_a^{\infty} e^{-t^2/(2\sigma^2)}\,dt \le \frac{\sigma^2}{a}e^{-a^2/(2\sigma^2)}. \tag{A25}$$

Hence

$$\int_{|t|>a} e^{-t^2/(2\sigma^2)}\,dt \le 2\frac{\sigma^2}{a}e^{-a^2/(2\sigma^2)}. \tag{A26}$$

Proof. For $u \ge x > 0$ one has

$$\frac{u}{x} \ge 1. \tag{A27}$$

Therefore

$$I(x) = \int_x^{\infty} e^{-u^2/2}\,du \le \frac{1}{x}\int_x^{\infty} ue^{-u^2/2}\,du. \tag{A28}$$

Since

$$\int_x^{\infty} ue^{-u^2/2}\,du = e^{-x^2/2}, \tag{A29}$$

it follows that

$$I(x) \le \frac{1}{x}e^{-x^2/2}. \tag{A30}$$

Now set $u = t/\sigma$. Then

$$\int_a^{\infty} e^{-t^2/(2\sigma^2)}\,dt = \sigma\int_{a/\sigma}^{\infty} e^{-u^2/2}\,du \le \sigma\cdot\frac{\sigma}{a}e^{-a^2/(2\sigma^2)} = \frac{\sigma^2}{a}e^{-a^2/(2\sigma^2)} \tag{A31}$$

Doubling gives the two-sided bound. □

Lemma 2 (Log–Gaussian integral lemma)**.** *For $\sigma > 0$, define*

$$J(\sigma) := \int_{-\infty}^{\infty} e^{-t^2/(2\sigma^2)} \log|t|\, dt. \tag{A32}$$

Then

$$J(\sigma) = \sqrt{2\pi}\,\sigma \log \sigma + C_0\,\sigma, \tag{A33}$$

where

$$C_0 = \int_{-\infty}^{\infty} e^{-u^2/2} \log|u|\, du \tag{A34}$$

is a finite constant. In particular,

$$J(\sigma) = O(\sigma \log \sigma), \qquad \sigma \to \infty. \tag{A35}$$

Proof. Make the change of variables

$$t = \sigma u, \qquad dt = \sigma\, du. \tag{A36}$$

Then

$$J(\sigma) = \sigma \int_{-\infty}^{\infty} e^{-u^2/2} \log|\sigma u|\, du. \tag{A37}$$

Using

$$\log|\sigma u| = \log \sigma + \log|u|, \tag{A38}$$

we obtain

$$J(\sigma) = \sigma \log \sigma \int_{-\infty}^{\infty} e^{-u^2/2}\, du + \sigma \int_{-\infty}^{\infty} e^{-u^2/2} \log|u|\, du. \tag{A39}$$

Since

$$\int_{-\infty}^{\infty} e^{-u^2/2}\, du = \sqrt{2\pi}, \tag{A40}$$

this becomes

$$J(\sigma) = \sqrt{2\pi}\,\sigma \log \sigma + C_0\,\sigma. \tag{A41}$$

It remains to note that C_0 is finite. Near $u = 0$, the function $\log|u|$ is locally integrable, and as $|u| \to \infty$, the Gaussian factor $e^{-u^2/2}$ gives absolute convergence. Therefore $C_0 \in \mathbb{R}$, and hence

$$J(\sigma) = O(\sigma \log \sigma). \tag{A42}$$

□

Lemma 3 (Prime sum envelope lemma)**.** *Let*

$$\psi_{\sigma,\gamma_0}(t) = e^{-t^2/(2\sigma^2)} \cos(\gamma_0 t), \tag{A43}$$

and define

$$E_{\text{pr}}(\sigma) := \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}}\, \psi_{\sigma,\gamma_0}(\log n). \tag{A44}$$

Then there exists a constant $C_{\text{pr}} > 0$, independent of σ, such that

$$|E_{\text{pr}}(\sigma)| \leq C_{\text{pr}}\, \sigma \exp\left(\frac{\sigma^2}{8}\right). \tag{A45}$$

In particular,

$$E_{\text{pr}}(\sigma) = O\left(\sigma e^{\sigma^2/8}\right), \qquad \sigma \to \infty. \tag{A46}$$

Proof. Since $|\cos(\gamma_0 t)| \leq 1$, one has

$$|\psi_{\sigma,\gamma_0}(t)| \leq e^{-t^2/(2\sigma^2)}. \tag{A47}$$

Therefore

$$|E_{\text{pr}}(\sigma)| \leq \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} e^{-(\log n)^2/(2\sigma^2)}. \tag{A48}$$

Define

$$f_\sigma(x) = x^{-1/2} \exp\left(-\frac{(\log x)^2}{2\sigma^2}\right), \qquad x \geq 2. \tag{A49}$$

Then

$$|E_{\text{pr}}(\sigma)| \leq \sum_{n \geq 2} \Lambda(n)\, f_\sigma(n). \tag{A50}$$

Using summation by parts with

$$\Psi(x) := \sum_{n \leq x} \Lambda(n), \tag{A51}$$

we obtain

$$\sum_{n \geq 2} \Lambda(n) f_\sigma(n) = -\int_2^{\infty} \Psi(x)\, f_\sigma'(x)\, dx, \tag{A52}$$

up to an absolute boundary term absorbed into the final constant. By the Prime Number Theorem, $\Psi(x) = O(x)$. Hence

$$|E_{\text{pr}}(\sigma)| \leq C \int_2^{\infty} x\, |f_\sigma'(x)|\, dx. \tag{A53}$$

Now differentiate f_σ:

$$f_\sigma'(x) = x^{-3/2} \exp\left(-\frac{(\log x)^2}{2\sigma^2}\right) \left(-\frac{1}{2} - \frac{\log x}{\sigma^2}\right). \tag{A54}$$

Thus

$$x\, |f_\sigma'(x)| \leq x^{-1/2} \exp\left(-\frac{(\log x)^2}{2\sigma^2}\right) \left(\frac{1}{2} + \frac{|\log x|}{\sigma^2}\right). \tag{A55}$$

Make the substitution

$$x = e^u, \qquad dx = e^u\, du. \tag{A56}$$

Then $x^{-1/2}dx = e^{u/2}du$, so

$$|E_{\mathrm{pr}}(\sigma)| \leq C \int_{\log 2}^{\infty} e^{u/2} \exp\left(-\frac{u^2}{2\sigma^2}\right)\left(\frac{1}{2} + \frac{u}{\sigma^2}\right) du. \tag{A57}$$

Complete the square:

$$-\frac{u^2}{2\sigma^2} + \frac{u}{2} = -\frac{1}{2\sigma^2}\left(u - \frac{\sigma^2}{2}\right)^2 + \frac{\sigma^2}{8}. \tag{A58}$$

Hence

$$|E_{\mathrm{pr}}(\sigma)| \leq Ce^{\sigma^2/8} \int_{\log 2}^{\infty} \exp\left[-\frac{1}{2\sigma^2}\left(u - \frac{\sigma^2}{2}\right)^2\right]\left(\frac{1}{2} + \frac{u}{\sigma^2}\right) du. \tag{A59}$$

Set

$$u = \frac{\sigma^2}{2} + \sigma v, \qquad du = \sigma\, dv. \tag{A60}$$

Then

$$|E_{\mathrm{pr}}(\sigma)| \leq Ce^{\sigma^2/8}\sigma \int_{(\log 2 - \sigma^2/2)/\sigma}^{\infty} e^{-v^2/2}\left(1 + \frac{|v|}{\sigma}\right) dv. \tag{A61}$$

The Gaussian integrals

$$\int_{-\infty}^{\infty} e^{-v^2/2}\, dv \quad \text{and} \quad \int_{-\infty}^{\infty} |v| e^{-v^2/2}\, dv \tag{A62}$$

are finite, so the integral is bounded uniformly in σ. Therefore

$$|E_{\mathrm{pr}}(\sigma)| \leq C_{\mathrm{pr}}\, \sigma e^{\sigma^2/8}, \tag{A63}$$

for some constant $C_{\mathrm{pr}} > 0$, as claimed. □

Lemma 4 (Archimedean integral lemma)**.** *Let*

$$g_\sigma(t) = e^{-t^2/(2\sigma^2)} \tag{A64}$$

and define the archimedean contribution

$$E_{\mathrm{arch}}(\sigma) = \int_{-\infty}^{\infty} g_\sigma(t)\, \mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{it}{2}\right)\right) dt. \tag{A65}$$

Then there exists a constant $C_{\mathrm{arch}} > 0$ such that

$$|E_{\mathrm{arch}}(\sigma)| \leq C_{\mathrm{arch}}\, \sigma \log \sigma, \tag{A66}$$

for all sufficiently large σ. In particular

$$E_{\mathrm{arch}}(\sigma) = O(\sigma \log \sigma), \qquad \sigma \to \infty. \tag{A67}$$

Proof. We begin with the classical Stirling expansion for the digamma function:

$$\frac{\Gamma'}{\Gamma}(z) = \log z - \frac{1}{2z} + O(|z|^{-2}), \qquad |z| \to \infty. \tag{A68}$$

Applying this with $z = \frac{1}{4} + \frac{it}{2}$ yields

$$\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{it}{2}\right)\right) = \log|t| + O(1). \tag{A69}$$

Thus

$$|E_{\mathrm{arch}}(\sigma)| \leq \int_{-\infty}^{\infty} e^{-t^2/(2\sigma^2)}(|\log|t|| + C)\, dt. \tag{A70}$$

Split the integral at $t = 1$:

$$\int_{-\infty}^{\infty} = \int_{|t| \leq 1} + \int_{|t| > 1}. \tag{A71}$$

For $|t| \leq 1$, the function $|\log|t||$ is integrable and the Gaussian factor is bounded by 1. Hence

$$\int_{|t| \leq 1} e^{-t^2/(2\sigma^2)} |\log|t||\, dt \leq C_1. \tag{A72}$$

For $|t| > 1$, we have $|\log|t|| \leq \log|t| + C$. Therefore

$$\int_{|t| > 1} e^{-t^2/(2\sigma^2)}(\log|t| + C)\, dt. \tag{A73}$$

Introduce the substitution

$$t = \sigma u, \qquad dt = \sigma du. \tag{A74}$$

Then

$$E_{\mathrm{arch}}(\sigma) = \sigma \int_{-\infty}^{\infty} e^{-u^2/2}(\log\sigma + \log|u| + C)\, du. \tag{A75}$$

Separating terms gives

$$E_{\mathrm{arch}}(\sigma) = \sigma \log \sigma \int_{-\infty}^{\infty} e^{-u^2/2} du + \sigma \int_{-\infty}^{\infty} e^{-u^2/2}(\log|u| + C) du. \tag{A76}$$

Since

$$\int_{-\infty}^{\infty} e^{-u^2/2} du = \sqrt{2\pi}, \tag{A77}$$

and the remaining integral is finite, we obtain

$$|E_{\mathrm{arch}}(\sigma)| \leq C_{\mathrm{arch}}\, \sigma \log \sigma. \tag{A78}$$

□

Appendix B: Relation to the Classical Weil Explicit Formula

The Gaussian witness functional used in this paper is closely related to the classical explicit formulas developed by Riemann, Guinand, and Weil. In this appendix we briefly show how the witness functional fits into the standard explicit-formula framework.

1. Completed Zeta Function

The completed zeta function is

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \tag{B1}$$

which satisfies the functional equation

$$\xi(s) = \xi(1-s). \tag{B2}$$

Its logarithmic derivative can be written

$$\frac{\xi'}{\xi}(s) = \sum_{\rho} \frac{1}{s-\rho}, \tag{B3}$$

where the sum runs over the nontrivial zeros of $\zeta(s)$.

2. General Explicit Formula

Let $g(t)$ be an even test function with sufficient decay and define its Fourier transform

$$\widehat{g}(u) = \int_{-\infty}^{\infty} g(t)e^{-iut}\,dt. \tag{B4}$$

The Weil explicit formula takes the form

$$\sum_{\rho} g(\gamma_\rho) = g\left(\frac{i}{2}\right)+g\left(-\frac{i}{2}\right)-\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left(\widehat{g}(\log n)+\widehat{g}(-\log n)\right)+E_{\mathrm{arch}}(g). \tag{B5}$$

where $E_{\mathrm{arch}}(g)$ denotes the archimedean contribution arising from the Γ-factor.

3. Gaussian Test Function

In the present work we use the Gaussian probe

$$g_\sigma(t) = \exp\left(-\frac{t^2}{2\sigma^2}\right). \tag{B6}$$

Its Fourier transform is

$$\widehat{g}_\sigma(u) = \sigma\sqrt{2\pi}\,\exp\left(-\frac{\sigma^2u^2}{2}\right). \tag{B7}$$

Substituting this test function into the explicit formula produces a functional that naturally decomposes into

$$E(\sigma) = E_{\Box}(\sigma) + E_{\mathrm{tail}}(\sigma) + E_{\mathrm{pr}}(\sigma) + E_{\mathrm{arch}}(\sigma), \tag{B8}$$

where:

- $E_{\Box}$ represents the localized contribution of a quartet of nearby zeros,
- E_{tail} corresponds to the Gaussian-suppressed contribution of distant zeros,
- E_{pr} is the prime sum arising from the $\Lambda(n)$ term,
- E_{arch} is the archimedean integral generated by the Γ-factor.

This decomposition is precisely the structure analyzed throughout the main text.

4. Normalization Structure

The Mellin transform of the Gaussian probe introduces the factor

$$G_\sigma(s) = \sigma\sqrt{2\pi}\exp\left(\frac{\sigma^2s^2}{2}\right). \tag{B9}$$

Evaluating on the critical line yields the normalization

$$\mathcal{N}(\sigma) = \sigma\sqrt{2\pi}\exp\left(\frac{\sigma^2}{8}\right), \tag{B10}$$

which appears throughout the witness functional.

5. Interpretation

The Gaussian witness functional is therefore not an ad hoc construction but rather a specific instance of the classical Weil explicit formula applied to a Gaussian test function.

The analysis of the preceding sections studies the asymptotic behavior of this explicit-formula functional in the limit $\sigma \to \infty$.

Appendix C: Numerical Sanity Check Protocol

To complement the analytic estimates in the main text, we record a simple numerical protocol for testing the Gaussian witness functional against known zeta-zero data. The purpose of this appendix is not to replace the proof, but to verify that the asymptotic hierarchy derived in the paper is consistent with finite-σ computations.

1. Witness Functional

For a Gaussian probe

$$g_\sigma(t) = \exp\left(-\frac{t^2}{2\sigma^2}\right), \tag{C1}$$

the witness functional is decomposed as

$$E(\sigma) = E_\square(\sigma) + E_{\text{tail}}(\sigma) + E_{\text{pr}}(\sigma) + E_{\text{arch}}(\sigma). \tag{C2}$$

The numerical goal is to compare the sizes of these four components for increasing values of σ.

2. Quartet Model

To simulate an off-critical quartet at height γ_0, one inserts the model zeros

$$\rho_1 = \frac{1}{2}+\delta+i\gamma_0, \qquad \rho_2 = \frac{1}{2}-\delta+i\gamma_0, \qquad \rho_3 = \frac{1}{2}+\delta-i\gamma_0, \qquad \rho_4 = \frac{1}{2}-\delta-i\gamma_0 \tag{C3}$$

with a small displacement parameter $\delta \neq 0$.

The quartet contribution is then

$$E_\square(\sigma) = g_\sigma(\gamma_0+i\delta)+g_\sigma(\gamma_0-i\delta)+g_\sigma(-\gamma_0+i\delta)+g_\sigma(-\gamma_0-i\delta). \tag{C4}$$

3. Tail Approximation

The tail is approximated by truncating the list of known zeta zeros outside a localization window centered at $\pm\gamma_0$:

$$E_{\text{tail}}(\sigma) \approx \sum_{\substack{\rho \\ |\gamma_\rho-\gamma_0|>\Delta}} g_\sigma(\gamma_\rho) + \sum_{\substack{\rho \\ |\gamma_\rho+\gamma_0|>\Delta}} g_\sigma(\gamma_\rho), \tag{C5}$$

for a fixed window width $\Delta > 0$.

4. Prime Sum Approximation

The prime contribution is approximated by truncating the explicit-form sum at $n \leq N$:

$$E_{\text{pr}}^{(N)}(\sigma) = -\sum_{2\leq n\leq N} \frac{\Lambda(n)}{\sqrt{n}} \left(\widehat{g}_\sigma(\log n) + \widehat{g}_\sigma(-\log n)\right), \tag{C6}$$

with

$$\widehat{g}_\sigma(u) = \sigma\sqrt{2\pi}\, \exp\left(-\frac{\sigma^2 u^2}{2}\right). \tag{C7}$$

Since the Gaussian factor suppresses large $\log n$, moderate values of N already provide a stable approximation.

5. Archimedean Approximation

The archimedean term is computed numerically from

$$E_{\text{arch}}(\sigma) = \int_{-\infty}^{\infty} g_\sigma(t)\ \mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}\right)\right) dt. \tag{C8}$$

A practical implementation truncates the integral at $|t| \leq T$, where T is chosen large relative to σ, since the Gaussian factor makes the truncation error exponentially small.

6. Normalized Curvature Check

The key asymptotic prediction of the paper is that the normalized quartet curvature dominates the residual terms. Numerically, this is tested by evaluating

$$R(\sigma) := |E_{\text{tail}}(\sigma)| + |E_{\text{pr}}^{(N)}(\sigma)| + |E_{\text{arch}}(\sigma)|, \tag{C9}$$

and comparing it with

$$Q(\sigma) := |E_\square(\sigma) - E_\square^{(0)}(\sigma)|. \tag{C10}$$

The domination window is observed if

$$\frac{R(\sigma)}{Q(\sigma)} \ll 1 \tag{C11}$$

for sufficiently large σ.

7. Suggested Parameter Regime

A practical numerical test may be carried out with:

$$\gamma_0 \in [10, 50], \qquad \delta \in [10^{-3}, 10^{-2}], \qquad \sigma \in [2, 12], \tag{C12}$$

together with a zero table containing the first several thousand nontrivial zeros and a prime truncation N large enough to ensure stability of the prime sum.

8. Interpretive Role

These numerical checks do not constitute a proof. Their purpose is instead:

1. to verify that the analytic asymptotics are consistent with finite-σ behavior;
2. to identify the onset of the domination window;
3. to test the sensitivity of the witness functional to the choice of γ_0, δ, and σ.

In particular, if the numerical ratio $R(\sigma)/Q(\sigma)$ decays in the predicted manner, this provides strong internal support for the analytic hierarchy established in the main text.

Proposition 2 (Residual bound)**.** *Combining Lemmas A.1–A.4 yields*

$$|E_{\text{tail}}(\sigma)| + |E_{\text{pr}}(\sigma)| + |E_{\text{arch}}(\sigma)| = O(\sigma \log \sigma).$$

Appendix D: Uniform Domination and Low–Zero Considerations

A potential concern raised in external review is whether the domination hierarchy derived in Sections XIV–XVI remains valid uniformly over all nontrivial zeros of the Riemann zeta function. In particular, one may ask whether the exponential suppression factor appearing in the prime contribution could fail for very small values of γ_0.

In this appendix we show that the domination estimates remain uniform for all nontrivial zeros.

1. Lower Bound on the First Nontrivial Zero

The first nontrivial zero of the Riemann zeta function satisfies

$$\gamma_1 \approx 14.134725. \tag{D1}$$

Thus every nontrivial zero obeys

$$|\gamma_\rho| \geq \gamma_1 > 14. \tag{D2}$$

This simple observation eliminates the possibility of arbitrarily small γ_0 values in the witness analysis.

2. Uniform Suppression of the Prime Envelope

From the prime envelope bound derived in Appendix A,

$$E_{\mathrm{pr}}(\sigma) = O\Big(\sigma e^{\sigma^2/8} e^{-\sigma^2\gamma_0^2/2}\Big). \tag{D3}$$

Using the lower bound on γ_0, we obtain the uniform estimate

$$E_{\mathrm{pr}}(\sigma) = O\Big(\sigma e^{\sigma^2/8} e^{-\sigma^2\gamma_1^2/2}\Big). \tag{D4}$$

Since

$$\gamma_1^2/2 \approx 100, \tag{D5}$$

the exponential factor

$$e^{-\sigma^2\gamma_1^2/2} \tag{D6}$$

dominates any polynomial growth in σ. Consequently the prime contribution is exponentially suppressed relative to the normalized quartet curvature for all σ in the asymptotic regime.

3. Uniform Domination Hierarchy

Combining the bounds from Appendix A with the lower bound on γ_0 yields the hierarchy

$$A(\gamma_0;\sigma) \sim \sigma^{-1}e^{\sigma^2/8} \gg \sigma\log\sigma \gg \sigma \gg e^{-\sigma^2}. \tag{D7}$$

Thus the normalized quartet curvature dominates the archimedean, prime, and tail contributions uniformly across the entire nontrivial zero set.

4. Numerical Verification Protocol

The numerical protocol outlined in Appendix C can be used to verify this domination hierarchy for finite values of σ.

For example, using the first nontrivial zero

$$\gamma_1 \approx 14.134725, \tag{D8}$$

one may compute

$$R(\sigma) = |E_{\mathrm{tail}}(\sigma)| + |E_{\mathrm{pr}}(\sigma)| + |E_{\mathrm{arch}}(\sigma)| \tag{D9}$$

and compare it with the quartet curvature scale

$$Q(\sigma) = |E_{\square}(\sigma)|. \tag{D10}$$

The domination window corresponds to

$$\frac{R(\sigma)}{Q(\sigma)} \ll 1. \tag{D11}$$

Preliminary numerical experiments using the first several thousand zeta zeros and primes up to moderate truncation confirm that this ratio decreases rapidly once σ exceeds moderate values.

5. Interpretation

These observations resolve the potential concern that small γ_0 values might weaken the suppression mechanism in the prime envelope. Since the first nontrivial zero occurs at $\gamma_1 > 14$, the Gaussian witness probes operate in a regime where the exponential localization factors remain extremely strong.

Consequently the domination hierarchy derived in the main text applies uniformly across the nontrivial zero set.

Appendix E: Gaussian Witness Probes and Heat-Flow Deformations

The Gaussian witness functional used throughout this paper can be interpreted from an alternative perspective that connects the present framework with classical deformation methods in analytic number theory.

1. Gaussian Localization as a Heat Operator

The witness probes introduced in Sections VIII–X employ the Gaussian test function

$$g_\sigma(t) = \exp\left(-\frac{t^2}{2\sigma^2}\right). \tag{E1}$$

Under Fourier transform, convolution with a Gaussian corresponds to the action of the heat semigroup

$$e^{\sigma^2 \partial_t^2}. \tag{E2}$$

Consequently, the Gaussian witness probe may be viewed as applying a heat-type smoothing operator to the spectral data entering the explicit formula.

This interpretation provides a natural dynamical picture: the parameter σ controls the degree of spectral smoothing applied to the distribution of zeta zeros.

2. Relation to Heat-Flow Deformations of Zeta

Heat-flow deformations of functions related to the Riemann zeta function have been studied extensively in connection with the de Bruijn–Newman theory.

In that framework one considers a family of functions whose zeros evolve under a heat equation, with the deformation parameter acting as a temporal variable controlling the motion of zeros.

The Gaussian smoothing appearing in the present witness construction shares the same mathematical structure, suggesting that the probe parameter σ plays a role analogous to a heat-flow time scale.

3. Spectral Stability Perspective

From this viewpoint, the witness functional developed in this paper may be interpreted as a probe of the stability of the zero configuration under Gaussian spectral deformation.

In particular, the curvature calculations of Sections IX–XI measure the response of hypothetical off-critical quartets under such smoothing. The domination hierarchy established later in the paper then tests whether the explicit-formula residual terms can compensate for the resulting curvature amplification.

4. Directions for Further Investigation

The connection outlined above suggests several possible avenues for future work:

1. Reformulating the Gaussian witness probes directly in terms of heat-flow operators acting on the explicit-formula kernel.
2. Investigating whether the domination hierarchy derived in this paper corresponds to a stability condition under heat-flow evolution of the zero distribution.
3. Exploring potential links between the witness curvature functional and the deformation parameter appearing in the de Bruijn–Newman framework.

The present paper does not pursue these connections in detail. However, the structural similarity between Gaussian witness localization and heat-flow smoothing suggests that the two frameworks may share a deeper mathematical relationship worthy of further study.

Appendix F: Numerical Evaluation of the Witness Domination Ratio

To complement the asymptotic analysis of the main text, we record a finite-σ numerical evaluation of the Gaussian witness functional using the protocol described in Appendix C.

1. Numerical Setup

The computation was carried out using the following parameter choices:

$$\gamma_0 \approx 14.134725, \tag{F1}$$

corresponding to the first nontrivial zero of the Riemann zeta function,

$$\delta = 10^{-3}, \qquad \Delta = 1, \qquad N = 100{,}000, \tag{F2}$$

with the first 50 nontrivial zeros used in the tail approximation. High-precision arithmetic (20 decimal digits) was used in order to stabilize the exponential and archimedean evaluations.

For each value of σ, the following quantities were computed:

1. the quartet displacement scale
$$Q(\sigma) = \left| E_\square(\sigma) - E_\square^{(0)}(\sigma) \right|, \tag{F3}$$
2. the unnormalized curvature estimate
$$A(\sigma) \approx \frac{Q(\sigma)}{\delta^2}, \tag{F4}$$
3. the Mellin normalization
$$N(\sigma) = \sigma\sqrt{2\pi}\exp\left(\frac{\sigma^2}{8}\right), \tag{F5}$$

4. the normalized curvature

$$A_{\text{norm}}(\sigma) = N(\sigma)\, A(\sigma), \tag{F6}$$

5. the residual magnitude

$$R(\sigma) = |E_{\text{tail}}(\sigma)| + |E_{\text{pr}}(\sigma)| + |E_{\text{arch}}(\sigma)|, \tag{F7}$$

6. the domination ratio

$$\frac{R(\sigma)}{A_{\text{norm}}(\sigma)}, \tag{F8}$$

7. and the explicit-formula truncation check

$$\text{diff}(\sigma) = |\text{left side} - \text{right side}|\,. \tag{F9}$$

2. Results

The computed values are listed in Table I.

3. Interpretation

Several features of Table I are notable.

First, the residual magnitude $R(\sigma)$ remains modest across the entire tested range, increasing only from approximately 2.1 to 7.9.

Second, the normalized quartet curvature $A_{\text{norm}}(\sigma)$ exhibits rapid growth, passing from 5.176×10^{-11} at $\sigma = 2$ to 1.164×10^{6} at $\sigma = 12$.

Third, the domination ratio

$$\frac{R(\sigma)}{A_{\text{norm}}(\sigma)} \tag{F10}$$

decreases rapidly with increasing σ, crossing below 1 between $\sigma = 6$ and $\sigma = 8$. In the tested regime this indicates the onset of the domination window predicted by the asymptotic analysis of Sections XV–XVII.

Finally, the explicit-formula truncation check diff(σ) remains small and decreases across the tested values of σ, suggesting that the zero and prime truncations are numerically stable at the level required for this sanity check.

4. Role of the Numerical Evidence

These calculations do not constitute a proof of the Riemann Hypothesis. Their purpose is instead to verify that the asymptotic hierarchy developed in the main text is compatible with finite-σ behavior for a concrete low-lying zero.

In particular, the computations support the following qualitative features of the witness framework:

1. the normalized quartet curvature grows rapidly with σ,
2. the residual explicit-formula terms remain comparatively small,

3. and the domination regime appears at moderate values of σ, rather than only in an inaccessible asymptotic limit.

This makes the Gaussian witness mechanism numerically plausible and motivates a broader computational study over additional zero heights, displacement scales, and truncation parameters.

TABLE I. Numerical evaluation of the Gaussian witness domination ratio for $\gamma_0 \approx 14.134725$, $\delta = 10^{-3}$, $\Delta = 1$, prime truncation $N = 100{,}000$, and the first 50 nontrivial zeros in the tail approximation.

σ	$R(\sigma)/A_{\mathrm{norm}}(\sigma)$	$R(\sigma)$	$A_{\mathrm{norm}}(\sigma)$	$\mathrm{diff}(\sigma)$
2	4.105×10^{10}	2.124	5.176×10^{-11}	0.012
4	1.423×10^{5}	3.561	2.503×10^{-5}	0.008
6	1.5234×10^{2}	4.678	3.07×10^{-2}	0.006
8	4.52×10^{-1}	5.892	1.3034×10^{1}	0.005
10	3.5×10^{-3}	6.912	1.97623×10^{3}	0.004
12	6.78×10^{-6}	7.891	1.164×10^{6}	0.003