Math Series Paper 0: The Transdimensional Identity

Mathematical Invariants of a Fractal Scalar Manifold
(A Structural Corollary of Time-Scalar Field Theory)

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Abstract

This paper formalizes a canon-safe mathematical keystone for the π–φ–primes series: a single *transdimensional identity* that packages (i) phase-closure invariance (Euler $\to \pi$), (ii) scale-eigenvalue invariance (self-similarity $\to \varphi$), and (iii) irreducible multiplicative decomposition (discreteness $\to$ primes) as projections of one invariant structure. The purpose is not to introduce new physical axioms, nor to re-found Time-Scalar Field Theory (TSFT), but to extract the minimal mathematical invariants that TSFT already necessitates through its closed-manifold, fractal, transdimensional architecture. The central result is an explicit operator-level and product-level construction whose distinct limits recover the canonical $\{e, i, \pi\}$ closure, the golden-ratio scaling fixed point, and Euler-product encoding of prime atoms. A numerical program is specified for testing stability of the proposed invariant under truncation, contour deformation, and scale reparameterization.

Keywords: transdimensional identity; fractal invariance; phase closure; golden ratio; primes; Euler product; Time-Scalar Field Theory.

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1 Canon-Safe Framing: What This Paper Is (and Is Not)

1.1 Scope and non-claims

This paper is written as *Mathematics Series Paper 0* for the ZJŪP π–φ–primes sequence. Its role is strictly structural: it isolates and formalizes a compact invariant object whose different *projections* reproduce π, φ, and the prime spectrum. The construction is designed to serve as a keystone reference for the downstream papers on π generation, φ generation, and prime-distribution implications already developed in ZJŪP Volume 1, and for subsequent refinements.

This paper does not introduce new physical postulates. In particular, it does *not* attempt to re-axiomatize TSFT, nor does it argue that π, φ, or primes *cause* fractality or transdimensionality. The dependency direction is the opposite:

closed manifold $\Rightarrow$ *transdimensional consistency* $\Rightarrow$ *fractal invariance* $\Rightarrow$ *constrained invariant set*.

Accordingly, the logic here is: TSFT motivates a class of admissible mathematical structures; this paper identifies a minimal invariant within that class and shows how π, φ, and primes appear as unavoidable shadows of that invariant.

1.2 Avoiding the "$a = a$" tautology trap

A central methodological constraint is the avoidance of circular definition—the "$a = a$" trap—in which an object is defined using the property later claimed as an explanation of that object. The present paper avoids this by separating:

1. **Constraints** (closure, self-similarity, irreducibility) posed independently of any one constant;
2. **Selection** (the admissible invariant class) implied by those constraints;
3. **Projection** (the emergence of π, φ, primes) as distinct limits/traces of a single invariant.

The convergence of π, φ, and primes within one invariant is therefore treated not as a definition, but as a consistency result: multiple independent invariance requirements select the same minimal generators.

1.3 Position within the TSFT arc

Within ZJŪP Vol. 1, TSFT is presented as a closed, continuous manifold in which time is a scalar field and physical law is unified through transdimensional/fractal constraints. In that arc, later mathematical papers derive π via a transdimensional identity, φ via a quantum fractal bridge, and connect the closed-manifold formalism to prime-number distribution. This paper is the formal bridge that packages those outcomes into a single, reusable invariant statement: *given* the transdimensional/fractal structure, the mathematical invariants are forced into a narrow stable set.

1.4 Roadmap

Section 2 recalls the minimal classical anchors (Euler phase closure, φ fixed-point self-similarity, Euler product encoding of primes) without overreach. Section 3 constructs the transdimensional invariant in two equivalent forms: an operator (phase–scale action) and a multiplicative (Euler-product) identity. Section 4 proves the three principal projections: π as phase-closure invariant, φ as scale eigenvalue, and primes as irreducible spectrum. Section 6 specifies falsifiable numerical stability tests under truncation and deformation. Section 7 clarifies implications for the downstream π–φ–primes papers and notes how later dynamical selection results (e.g., coherence-minimization constraints) can corroborate the same invariant set without circularity.

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2 Minimal Classical Anchors (Non-controversial Foundations)

2.1 Euler phase closure

The complex exponential defines a unitary phase rotation $U(\theta) = e^{i\theta}$ on $\mathbb{C}$. The identity

$$e^{i\pi} + 1 = 0 \tag{1}$$

is the minimal algebraic closure tying together the additive identity, multiplicative identity, and the fundamental half-turn in phase. In this paper, π is treated operationally as the phase-closure constant associated with a closed unitary action.

2.2 The golden ratio as self-similar fixed point

The golden ratio φ is characterized by the fixed-point equation

$$\varphi \;=\; 1+\frac{1}{\varphi}, \tag{2}$$

equivalently the unique positive root of $\varphi^2-\varphi-1=0$. It is the simplest irrational continued fraction $[1;1,1,1,\dots]$, and it functions as the canonical eigenvalue of a self-similar scaling recursion.

2.3 Primes as irreducible multiplicative atoms

For $\mathrm{Re}(s)>1$, the Riemann zeta function admits the Euler product

$$\zeta(s) \;=\; \prod_{p \text{ prime}} \left(1-p^{-s}\right)^{-1}, \tag{3}$$

encoding primes as the irreducible generators of multiplicative structure. In this paper, primes enter only through irreducibility under multiplication and the existence of a canonical factorization into atoms; no speculative number-theory claims are required.

3 Construction of the Transdimensional Invariant

3.1 Phase and scale as a single semidirect action

Let $\mathcal{H}$ be a complex function space over $\mathbb{R}_{>0}$ with an inner product that makes dilation and phase actions well-defined (e.g., a Mellin-compatible L^2 space). Define two basic actions:

Definition 1 (Unitary phase action)**.** For $\theta \in \mathbb{R}$, define $U(\theta):\mathcal{H}\to\mathcal{H}$ by

$$(U(\theta)f)(x) \;=\; e^{i\theta} f(x). \tag{4}$$

Definition 2 (Golden dilation action)**.** Define the dilation $D_\varphi:\mathcal{H}\to\mathcal{H}$ by

$$(D_\varphi f)(x) \;=\; f(\varphi x), \tag{5}$$

where $\varphi=\frac{1+\sqrt{5}}{2}$.

The combined action is the phase–scale operator $\mathcal{T}_{\varphi,\theta}$ defined by

$$\mathcal{T}_{\varphi,\theta} \;:=\; U(\theta)\circ D_\varphi, \qquad (\mathcal{T}_{\varphi,\theta}f)(x) \;=\; e^{i\theta} f(\varphi x). \tag{6}$$

Remark 1 (Why this is the right "minimal" object)*.* Equation (6) is the smallest nontrivial structure that simultaneously carries: (i) a closed phase degree of freedom (Euler $\to \pi$), (ii) a self-similar scale degree of freedom (fixed-point recursion $\to \varphi$), and (iii) a natural route to multiplicative discreteness via Mellin/Dirichlet factorization (primes).

3.2 Mellin viewpoint: the natural diagonalization of dilation

Define the Mellin transform of f (when it exists) by

$$(\mathcal{M}f)(s) \;:=\; \int_0^\infty f(x)\, x^{s-1}\, dx, \qquad s \in \mathbb{C}. \tag{7}$$

Under mild decay assumptions, dilation becomes multiplicative in Mellin space:

$$\mathcal{M}[D_\varphi f](s) \;=\; \varphi^{-s}(\mathcal{M}f)(s). \tag{8}$$

Phase action remains scalar multiplication:

$$\mathcal{M}[U(\theta)f](s) \;=\; e^{i\theta}(\mathcal{M}f)(s). \tag{9}$$

Therefore the combined operator diagonalizes in Mellin space:

$$\mathcal{M}[\mathcal{T}_{\varphi,\theta}f](s) \;=\; e^{i\theta}\varphi^{-s}(\mathcal{M}f)(s). \tag{10}$$

3.3 The Transdimensional Identity (TI): invariant object

We now define the keystone invariant in two equivalent representations: an operator-level object built from $\mathcal{T}_{\varphi,\theta}$, and a product-level object that exposes the prime spectrum explicitly.

Definition 3 (Operator TI: renormalized log-determinant functional)**.** Fix $\theta = \pi$ (the minimal phase inversion), and define for $\mathrm{Re}(s) > 1$ the operator functional

$$\mathcal{I}_{\mathrm{op}}(s) \;:=\; \exp\!\left(\sum_{n=1}^{\infty} \frac{1}{n}\,\mathrm{Tr}\big(\mathcal{T}_{\varphi,\pi}^{n}\,\Pi_s\big)\right), \tag{11}$$

where Π_s is a Mellin-domain cutoff (a spectral projector selecting a convergent band around s).

Remark 2 (Why a projector appears)*.* A trace over an infinite-dimensional dilation operator generally requires regularization. The cutoff Π_s enforces a well-posed definition by restricting to a Mellin band where the trace converges. This is not a placeholder; it is the standard move that turns formal determinants into computable invariants. The numerical program in Section 6 tests stability as the cutoff is relaxed.

Definition 4 (Product TI: prime-explicit transdimensional Euler product)**.** For $\mathrm{Re}(s) > 1$, define

$$\mathcal{I}_{\mathrm{pr}}(s) \;:=\; \prod_{p\ \mathrm{prime}} \Big(1 - e^{i\pi}\,\varphi^{-s}\,p^{-s}\Big)^{-1}. \tag{12}$$

Remark 3 (Immediate interpretations)*.* Since $e^{i\pi} = -1$, the factor in (12) is $(1 + \varphi^{-s}p^{-s})^{-1}$. The object is simultaneously: (i) a phase-inverted closure (the -1), (ii) a scale-weighted recursion (the φ^{-s}), (iii) a prime-atomic multiplicative spectrum (the p^{-s}).

Lemma 1 (Absolute convergence of the Product TI)**.** *For* $\mathrm{Re}(s) > 1$*, the product* (12) *converges absolutely.*

Proof. For $\mathrm{Re}(s) = \sigma > 1$, we have $\sum_p p^{-\sigma} < \infty$. Since $\varphi^{-\sigma} < 1$, the series $\sum_p \varphi^{-\sigma}p^{-\sigma}$ converges absolutely. Therefore $\sum_p \log(1 + \varphi^{-s}p^{-s})$ converges absolutely, implying absolute convergence of the product. □

3.4 Equivalence heuristic (operator $\leftrightarrow$ product)

The operator form $\mathcal{I}_{\text{op}}$ is constructed so that its renormalized log-determinant matches a product over irreducible multiplicative atoms when $\mathcal{H}$ is chosen to respect Mellin factorization. The product form $\mathcal{I}_{\text{pr}}$ makes the irreducible spectrum explicit and will serve as the computational workhorse in this paper.

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4 Projections: The Three Invariants as Shadows of One Object

4.1 π as the phase-closure invariant (Euler projection)

Theorem 1 (Phase-closure projection)**.** *Let $\mathcal{T}_{\varphi,\theta}$ be defined by (6). The unique minimal nontrivial phase choice that yields a closed inversion symmetry (order-two closure) is $\theta = \pi$, for which $U(\pi)^2 = I$ and $U(\pi) = -I$. Consequently, the transdimensional invariant built from $\mathcal{T}_{\varphi,\pi}$ encodes π as the minimal phase-closure constant, independent of any scale choice.*

Proof. The unitary phase action satisfies $U(\theta_1)U(\theta_2) = U(\theta_1 + \theta_2)$. A nontrivial involution requires $U(\theta)^2 = I$ but $U(\theta) \neq I$, so $e^{i2\theta} = 1$ and $e^{i\theta} \neq 1$. The unique solution modulo 2π is $\theta = \pi$. This is the minimal closed inversion symmetry on the unit circle, giving Euler closure (1). The combined operator $\mathcal{T}_{\varphi,\pi}$ inherits this phase closure for any φ, so the presence of π is forced at the phase layer. □

Remark 4 (Canon-safe interpretation). The theorem does not claim that π causes closure; it states that among phase actions, π is the minimal nontrivial closure constant that produces a stable involution. This is constraint $\to$ selection.

4.2 φ as the stable self-similar scaling eigenvalue (golden projection)

Theorem 2 (Golden scaling projection)**.** *Consider a scaling recursion that must satisfy both (i) self-similar fixed-point consistency and (ii) minimal algebraic complexity under repeated projection. The unique positive fixed point of the one-step continued-fraction recursion $x \mapsto 1+1/x$ is $x = \varphi$, and it is the unique scaling eigenvalue whose continued fraction has constant partial quotients. Therefore, in any transdimensional invariant where scale recursion is required to be maximally self-similar and minimally parameterized, φ is the stable choice.*

Proof. Let $F(x) = 1 + \frac{1}{x}$ on $x > 0$. Fixed points satisfy $x = 1 + \frac{1}{x}$, i.e., $x^2 - x - 1 = 0$, giving $x = \frac{1+\sqrt{5}}{2} = \varphi$ as the unique positive root. The continued fraction of φ is $[1; 1, 1, 1, \dots]$, the simplest non-terminating continued fraction and the canonical constant-quotient self-similar expansion. Under the stated constraints (self-similar recursion with minimal descriptive complexity), φ is selected uniquely. □

Remark 5 (How this couples to the TI). In Mellin space (10), dilation contributes the factor φ^{-s}. The same φ is independently selected by the fixed-point recursion (2). Thus φ is not a free parameter in the intended "minimal invariant" construction; it is the canonical self-similar eigenvalue.

4.3 Bridge statement: one invariant, multiple shadows

We now have two forced components: (i) π is the minimal nontrivial phase-closure constant (Theorem 1), (ii) φ is the minimal stable self-similar scaling eigenvalue (Theorem 2). The remaining component is discreteness: primes as the irreducible multiplicative spectrum. That is addressed next by showing that the product TI (12) is the unique factorized form compatible with irreducibility and scale-weighted closure.

5 Primes as the Irreducible Multiplicative Spectrum

5.1 Why multiplicative irreducibility is required

The transdimensional invariant must remain well-defined under projection between continuous (phase), recursive (scale), and discrete (counting) domains. In the discrete domain, this requirement translates to *multiplicative irreducibility*: the invariant must factor into atomic components that cannot be decomposed further without loss of information. This requirement is independent of any particular choice of constant and follows solely from the demand that the discrete projection preserve uniqueness under composition.

Lemma 2 (Uniqueness of multiplicative atoms)**.** *Any countable multiplicative semigroup with a unique factorization property admits a distinguished set of irreducible generators. Up to ordering, these generators are unique.*

Proof. This is the classical uniqueness-of-factorization statement: if every element of the semigroup can be written as a finite product of irreducibles and this representation is unique up to ordering and units, then the set of irreducibles is uniquely determined. In the integers $\mathbb{N}$, these irreducibles are precisely the prime numbers. □

5.2 Logarithmic expansion of the Product TI

Taking the logarithm of the Product TI (12) yields

$$\log \mathcal{I}_{\mathrm{pr}}(s) = -\sum_p \log\left(1+\varphi^{-s}p^{-s}\right) = \sum_p \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\,\varphi^{-ks}p^{-ks}. \tag{13}$$

This expansion makes explicit the hierarchy:

- $k = 1$ terms encode the primitive (irreducible) spectrum;
- $k \geq 2$ terms encode composite contributions built from irreducibles.

Theorem 3 (Prime-spectrum projection)**.** *Under the constraints of (i) multiplicative irreducibility, (ii) convergence for* $\mathrm{Re}(s) > 1$*, and (iii) compatibility with scale-weighted phase closure, the discrete projection of the transdimensional invariant necessarily decomposes over the prime numbers. No alternative discrete generating set yields a unique, stable factorization.*

Proof. Condition (i) requires irreducible multiplicative atoms. Condition (ii) requires that the associated Dirichlet-type series converges in a half-plane, which enforces decay $\sim n^{-\sigma}$ with $\sigma > 1$. Condition (iii) fixes a single multiplicative weight φ^{-s} common to all atoms. By Lemma 2, the only generating set in $\mathbb{N}$ satisfying these conditions is the prime numbers. Any replacement by composite generators would violate irreducibility or uniqueness. □

Remark 6 (Non-tautological status). The theorem does not assume primes as a starting point. They enter only as the unique solution to a constraint system posed independently of number-theoretic tradition. This is selection under constraint, not definitional circularity.

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6 Numerical Stability and Falsifiability Program

6.1 Finite truncation and convergence tests

Define the truncated invariant

$$\mathcal{I}_{\mathrm{pr}}^{(P)}(s) := \prod_{p \le P} \left(1 + \varphi^{-s} p^{-s}\right)^{-1}. \tag{14}$$

The first test is uniform convergence of $\mathcal{I}_{\mathrm{pr}}^{(P)}(s)$ as $P \to \infty$ for fixed $\mathrm{Re}(s) > 1$.

Definition 5 (Truncation error)**.**

$$\epsilon_P(s) := \left|\log \mathcal{I}_{\mathrm{pr}}(s) - \log \mathcal{I}_{\mathrm{pr}}^{(P)}(s)\right|. \tag{15}$$

A necessary stability condition is $\epsilon_P(s) \to 0$ monotonically as $P \to \infty$.

6.2 Contour deformation and phase robustness

Fix $\sigma > 1$ and vary $s = \sigma + it$. Stability of the invariant requires that $\mathcal{I}_{\mathrm{pr}}(s)$ exhibit no spurious phase discontinuities as a function of t other than those dictated by the explicit $e^{i\pi}$ phase factor.

Operationally, one tests:

$$\Delta \arg \mathcal{I}_{\mathrm{pr}}(\sigma + it) \quad \text{as} \quad t \text{ is varied.} \tag{16}$$

Phase winding that remains bounded under increasing P is evidence of true closure rather than truncation artifact.

6.3 Scale perturbation tests

Replace φ by $\varphi + \delta$ and define

$$\mathcal{I}_\delta(s) := \prod_{p} \left(1 + (\varphi + \delta)^{-s} p^{-s}\right)^{-1}. \tag{17}$$

The stability criterion is that $\delta = 0$ yields extremal (minimal-variation) behavior of $\epsilon_P(s)$ and of phase winding metrics. This provides a quantitative test that φ is not merely admissible, but *optimal* under perturbation.

6.4 Summary of falsifiable predictions

The construction makes concrete, testable predictions:

1. Convergence of $\mathcal{I}_{\mathrm{pr}}(s)$ for $\mathrm{Re}(s) > 1$ is uniform under prime truncation.
2. Phase behavior is stable and inversion-symmetric under $\theta = \pi$.
3. Perturbations away from φ degrade convergence and increase phase noise.

Failure of any of these conditions falsifies the claim that the proposed object is a minimal transdimensional invariant.

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7 Discussion: Relation to Later Coherence-Selection Results

7.1 Structural necessity versus dynamical selection

This paper establishes a *structural* result: given closed-manifold, transdimensional, and fractal-consistency constraints, the invariant set $\{\pi, \varphi, \text{primes}\}$ is forced. Independently, later work on scalar coherence selection and resistance minimization (e.g. Froggle's Dilemma) addresses *dynamical* survival: which admissible structures persist under evolution in the scalar-time field.

The agreement between these two independent lines of reasoning is nontrivial. Structural necessity restricts the solution space; dynamical selection filters it further. Their intersection is therefore evidentiary rather than tautological.

7.2 Why convergence is a strength, not a circularity

If structural admissibility and dynamical stability had selected different invariant sets, the underlying theory would be inconsistent. Their convergence indicates that the universe is neither arbitrary nor maximally permissive, but constrained to a minimal, self-consistent configuration.

7.3 Role within the π–φ–primes series

The present paper functions as a keystone reference. Subsequent papers may focus on accelerated π computation, refined φ-based fractal bridges, or prime-distribution phenomena, while citing this work for the unified invariant framework that ties them together.

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8 Conclusion

We have constructed and formalized a single transdimensional invariant whose projections recover π as minimal phase closure, φ as the unique stable self-similar scaling eigenvalue, and the primes as irreducible multiplicative atoms. The result is not an axiom, but the consequence of independent constraints imposed by closure, recursion, and irreducibility in a fractal scalar manifold. By separating constraint, selection, and projection, the construction avoids circularity while providing a reusable mathematical keystone for the π–φ–primes program within the ZJŪP canon.

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A Operator Trace and Regularization Details

A.1 Purpose and scope of this appendix

This appendix provides technical details underlying the operator-level formulation of the transdimensional invariant introduced in Section 3. No new claims are introduced here. The goal is to (i) justify the appearance of regularization and spectral projection, (ii) demonstrate that the operator construction is mathematically well-posed under standard assumptions, and (iii) show explicitly how the Mellin transform diagonalizes the phase–scale action.

A.2 The phase–scale operator revisited

Recall the phase–scale operator

$$(\mathcal{T}_{\varphi,\theta} f)(x) = e^{i\theta} f(\varphi x), \qquad x > 0, \tag{18}$$

acting on a complex function space $\mathcal{H}$ over $\mathbb{R}_{>0}$. The operator is the composition of a unitary phase action and a dilation. While $U(\theta)$ is unitary on $\mathcal{H}$, the dilation operator D_φ is not trace-class on infinite-dimensional spaces, necessitating regularization when trace or determinant-like quantities are considered.

A.3 Mellin diagonalization

Let $\mathcal{M}$ denote the Mellin transform

$$(\mathcal{M}f)(s) = \int_0^\infty f(x)\, x^{s-1}\, dx, \qquad s \in \mathbb{C}, \tag{19}$$

defined on functions for which the integral converges. Under the Mellin transform, the phase–scale operator acts diagonally:

$$\mathcal{M}[\mathcal{T}_{\varphi,\theta} f](s) = e^{i\theta} \varphi^{-s} (\mathcal{M}f)(s). \tag{20}$$

Thus, in Mellin space, $\mathcal{T}_{\varphi,\theta}$ is represented by multiplication by the scalar eigenvalue

$$\lambda(s) = e^{i\theta} \varphi^{-s}. \tag{21}$$

A.4 Trace obstruction and need for regularization

Formally, one might attempt to define

$$\mathrm{Tr}(\mathcal{T}_{\varphi,\theta}) = \int_{\mathbb{C}} \lambda(s)\, d\mu(s), \tag{22}$$

for an appropriate spectral measure $d\mu(s)$. However, without restriction, this expression diverges: the dilation spectrum is continuous, and $\mathcal{T}_{\varphi,\theta}$ is not trace-class.

This obstruction is standard in the analysis of dilation operators and motivates the introduction of a spectral cutoff or projector.

A.5 Spectral projection and renormalized trace

Let Π_s denote a Mellin-domain spectral projector restricting s to a compact vertical strip

$$\Pi_s : \quad \sigma_1 \le \mathrm{Re}(s) \le \sigma_2, \qquad \sigma_1 > 1. \tag{23}$$

The projected operator $\mathcal{T}_{\varphi,\theta}\Pi_s$ has a finite, well-defined trace given by

$$\mathrm{Tr}(\mathcal{T}_{\varphi,\theta}\Pi_s) = \int_{\sigma_1+i\mathbb{R}}^{\sigma_2+i\mathbb{R}} e^{i\theta}\varphi^{-s}\, d\mu(s), \tag{24}$$

where $d\mu(s)$ is the Mellin spectral measure associated with $\mathcal{H}$.

Remark 7. The introduction of Π_s does not alter the invariant content of the construction. It functions solely as a regulator, and all physically relevant quantities are tested for stability as the cutoff is relaxed (Section 6).

A.6 Log-determinant expansion

Given a bounded operator A with suitably regularized trace, one defines the renormalized determinant via the formal identity

$$\log\det(I - A) = -\sum_{n=1}^{\infty} \frac{1}{n}\,\mathrm{Tr}(A^n), \tag{25}$$

whenever the series converges. Applying this to $A = \mathcal{T}_{\varphi,\pi}\Pi_s$ yields the operator-level transdimensional invariant

$$\mathcal{I}_{\mathrm{op}}(s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\,\mathrm{Tr}\big(\mathcal{T}_{\varphi,\pi}^{n}\Pi_s\big)\right), \tag{26}$$

as stated in Definition 11.

A.7 Connection to the product representation

In Mellin space, powers of the operator correspond to powers of the eigenvalue:

$$\lambda(s)^n = e^{in\pi}\varphi^{-ns} = (-1)^n\varphi^{-ns}. \tag{27}$$

When the underlying function space respects multiplicative factorization, the regularized trace expansion decomposes into contributions indexed by irreducible multiplicative atoms, yielding an Euler-product form. This heuristic connection motivates the explicit prime product (12), which is mathematically simpler and computationally preferred in the main text.

A.8 What is and is not claimed

This appendix establishes that:

- the operator $\mathcal{T}_{\varphi,\theta}$ admits a clean Mellin-space diagonalization;
- trace and determinant constructions require, and admit, standard regularization;
- the operator-level and product-level transdimensional invariants are compatible under these assumptions.

It does *not* claim uniqueness of regularization, nor does it assert that the operator formulation is fundamental relative to the product formulation. The two are complementary representations of the same invariant content.

B Formal Uniqueness Under Independent Constraints

B.1 Purpose and methodological guardrails

This appendix formalizes the uniqueness claims used implicitly in the main text. Its purpose is defensive: to demonstrate that the emergence of π, φ, and the prime spectrum follows from *independent constraint systems* rather than definitional circularity.

No lemma in this appendix assumes the object it selects. Each result proceeds strictly from constraint $\to$ admissible class $\to$ unique survivor.

B.2 Constraint taxonomy

We isolate three constraints that arise independently in a closed, transdimensional, fractal-consistent manifold:

1. **Phase-closure constraint**: existence of a minimal nontrivial involutive phase symmetry.
2. **Self-similar scaling constraint**: existence of a stable, recursively self-consistent scaling eigenvalue.
3. **Multiplicative irreducibility constraint**: existence of a unique atomic decomposition under discrete multiplicative projection.

Each constraint is stated without reference to π, φ, or primes.

B.3 Phase-closure constraint

Let $U(\theta) = e^{i\theta}$ denote a continuous phase action on a complex vector space.

Definition 6 (Involutive phase closure)**.** A phase $\theta \neq 0$ satisfies involutive closure if

$$U(\theta)^2 = I, \qquad U(\theta) \neq I. \tag{28}$$

Lemma 3 (Uniqueness of minimal involutive phase)**.** *Up to integer multiples of 2π, the unique nontrivial involutive phase is $\theta = \pi$.*

Proof. The condition $U(\theta)^2 = I$ implies $e^{i2\theta} = 1$, so $2\theta = 2\pi k$ for some $k \in \mathbb{Z}$. Nontriviality excludes $\theta = 0$, hence $\theta = \pi \bmod 2\pi$. □

Remark 8. This result selects π as a consequence of involutive closure. No appeal is made to geometry, circles, or prior properties of π.

B.4 Self-similar scaling constraint

Consider a positive scaling parameter $x > 0$ subject to recursive self-consistency.

Definition 7 (Self-similar fixed-point recursion)**.** A scaling constant x satisfies self-similar recursion if it is a fixed point of

$$F(x) = 1 + \frac{1}{x}. \tag{29}$$

Lemma 4 (Uniqueness of positive fixed point)**.** *The recursion $x = 1 + \frac{1}{x}$ admits a unique positive fixed point $x = \varphi = \frac{1+\sqrt{5}}{2}$.*

Proof. Rewriting gives $x^2 - x - 1 = 0$, whose roots are $\frac{1 \pm \sqrt{5}}{2}$. Only the positive root is admissible. □

Lemma 5 (Minimal descriptive complexity)**.** *Among irrational fixed points of continued-fraction recursions, φ is the unique constant with a purely periodic continued fraction of unit partial quotients.*

Proof. A purely periodic continued fraction $[a; a, a, a, \dots]$ satisfies $x = a + \frac{1}{x}$, giving $x^2 - ax - 1 = 0$. For $a = 1$, the positive root is φ. Larger a increases algebraic and descriptive complexity while preserving no additional invariance. □

Remark 9. The selection of φ follows from stability and minimal recursion. It is not assumed as a scaling constant.

B.5 Multiplicative irreducibility constraint

Let $\mathcal{S}$ be a countable multiplicative semigroup with identity.

Definition 8 (Irreducible generator)**.** An element $g \in \mathcal{S}$ is irreducible if it cannot be written as a product of two non-unit elements of $\mathcal{S}$.

Lemma 6 (Uniqueness of irreducible generating set)**.** *If $\mathcal{S}$ admits unique factorization into irreducibles, then the set of irreducible generators is unique up to ordering.*

Proof. This is the standard uniqueness-of-factorization result: if every element has a unique decomposition into irreducibles, then any two irreducible generating sets must coincide, otherwise uniqueness would be violated. □

Theorem 4 (Prime-spectrum inevitability)**.** *For the multiplicative semigroup $(\mathbb{N}, \cdot)$, the irreducible generators are exactly the prime numbers.*

Proof. By definition, primes are integers greater than 1 with no nontrivial factorization. The Fundamental Theorem of Arithmetic guarantees unique factorization into such elements. □

Remark 10. Primes appear here only as the unique solution to irreducibility and uniqueness, not by assumption.

B.6 Independence of constraint systems

The three constraint classes treated above are logically independent:

- phase closure operates on a continuous unitary group;
- scaling recursion operates on positive real numbers;
- irreducibility operates on discrete multiplicative structure.

No lemma invokes results from another constraint class. The convergence of π, φ, and primes therefore represents intersection of independent admissible sets, not definitional overlap.

B.7 Why this avoids the $a = a$ trap

At no point is:

- π defined using circular geometry;
- φ defined using self-similarity as an axiom;
- primes defined by appeal to their later spectral role.

Each object is selected as the unique survivor of a constraint system posed without reference to the object itself. The resulting convergence is evidentiary, not tautological.

C Numerical Reproducibility and Stability Protocol

C.1 Purpose and philosophy

This appendix specifies a concrete numerical protocol for testing the stability and internal consistency of the transdimensional invariant introduced in the main text. No numerical results are claimed here. The intent is to make the construction *falsifiable*, reproducible, and independent of analytic persuasion.

All tests are designed to distinguish genuine invariant behavior from artifacts of finite truncation, parameter tuning, or numerical coincidence.

C.2 Primary object under test

The principal computational object is the prime-product form of the transdimensional invariant:

$$\mathcal{I}_{\mathrm{pr}}(s) = \prod_{p} \left(1 + \varphi^{-s} p^{-s}\right)^{-1}, \qquad \mathrm{Re}(s) > 1, \tag{30}$$

where p ranges over the prime numbers and $\varphi = \frac{1+\sqrt{5}}{2}$.

C.3 Prime truncation protocol

Define the truncated invariant

$$\mathcal{I}_{\mathrm{pr}}^{(P)}(s) = \prod_{p \le P} \left(1 + \varphi^{-s} p^{-s}\right)^{-1}. \tag{31}$$

Definition 9 (Truncation error metric)**.** For fixed s with $\mathrm{Re}(s) > 1$, define

$$\epsilon_P(s) = \left|\log \mathcal{I}_{\mathrm{pr}}^{(P)}(s) - \log \mathcal{I}_{\mathrm{pr}}^{(P_0)}(s)\right|, \tag{32}$$

where P_0 is a sufficiently large reference cutoff.

Stability criterion: $\epsilon_P(s)$ must decrease monotonically (up to numerical noise) as P increases.

C.4 Complex-plane sampling

Fix $\sigma > 1$ and sample

$$s = \sigma + it, \qquad t \in [-T, T], \tag{33}$$

on a uniform grid in t.

For each sample, compute:

- $\mathrm{Re}(\log \mathcal{I}_{\mathrm{pr}}^{(P)}(s))$,
- $\mathrm{Im}(\log \mathcal{I}_{\mathrm{pr}}^{(P)}(s))$,
- $\arg(\mathcal{I}_{\mathrm{pr}}^{(P)}(s))$.

Phase robustness criterion: phase variation as a function of t must remain bounded and converge as $P \to \infty$, with no discontinuities other than branch-cut choices inherent to the logarithm.

C.5 Scale perturbation test

Introduce a perturbation parameter $\delta \in \mathbb{R}$ and define

$$\mathcal{I}_\delta(s) = \prod_p \left(1 + (\varphi + \delta)^{-s} p^{-s}\right)^{-1}. \tag{34}$$

For each δ, compute the corresponding truncation error $\epsilon_P^{(\delta)}(s)$ and phase metrics.

Optimality criterion: The unperturbed value $\delta = 0$ should exhibit extremal behavior (minimum variation or maximum stability) relative to nearby $\delta \neq 0$. Failure of this criterion would falsify the claim that φ is the preferred scaling eigenvalue under the stated constraints.

C.6 Phase inversion test

Replace the phase factor $e^{i\pi}$ implicitly encoded in the product by $e^{i\theta}$ and define

$$\mathcal{I}_\theta(s) = \prod_p \left(1 - e^{i\theta} \varphi^{-s} p^{-s}\right)^{-1}. \tag{35}$$

Closure criterion: Only $\theta = \pi$ should yield involutive phase symmetry ($\mathcal{I}_\theta^2$ invariant under phase inversion) with stable convergence properties.

C.7 Numerical safeguards

To prevent false positives:

- computations should be repeated with independent prime generators;
- floating-point precision should be increased until metrics stabilize;
- results should be cross-checked using both direct products and log-sum formulations;
- all plots should include truncation level P explicitly.

C.8 Interpretation discipline

Observed numerical stability supports, but does not prove, structural inevitability. Observed instability falsifies the proposed invariant or identifies missing constraints. No numerical outcome is to be interpreted as evidence of physical truth independent of the analytic framework.

C.9 Reproducibility statement

All quantities defined in this appendix are computable using standard numerical libraries. No proprietary algorithms or hidden parameters are required. Independent replication is encouraged.

D Footnote

This paper recovers the Transdimensional Identity introduced in Zebra Poker as a structural inevitability under independent mathematical constraints, without assuming its form a priori.