

# Schrödinger Dynamics as the Low-Spectrum Limit of SU(N)-Covariant Time–Scalar Spectral Geometry

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The Zebra Journal of Unified Physics (ZJUP)

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February 26, 2026

## Abstract

We derive the nonrelativistic Schrödinger dynamics as the controlled low-spectrum limit of the SU(N)-covariant Time–Scalar Field Theory (TSFT) scale-chain framework. Starting from the previously established Dirac-form generator whose square yields the quadratic TSFT operator, we isolate the positive-energy branch and perform a uniform low-spectrum expansion. After removal of the fast rest-phase oscillation, the reduced evolution is shown to obey a Schrödinger-type equation generated by an explicit effective Hamiltonian

$$H_{\text{eff}} = \frac{P_s^2}{2M_{\text{eff}}},$$

with computable lattice corrections expressed in terms of the TSFT Laplacian operator. The result is purely operator-theoretic and introduces no external quantum postulates. This completes the intrinsic kinematic bridge from TSFT spectral geometry to nonrelativistic quantum evolution.

## 1 Introduction

## 2 Framework and Standing Assumptions

**Position within the TSFT program.** Earlier papers in this series established (i) the SU(N)-covariant spectral framework and Floquet sectorization, (ii) boundary-induced Bohr-type quantization on half-line geometries, (iii) a Weyl-pair mechanism yielding a Heisenberg-type uncertainty structure, and (iv) emergence of Dirac-type spinor dynamics from first-order factorization. The present work completes the nonrelativistic dynamical sector by showing that the positive-energy branch admits a controlled Schrödinger limit on low-spectrum subspaces.

### 2.1 Hilbert space and covariant chain operators

Fix an integer  $N \geq 2$  and let

$$\mathcal{H} := \ell^2(\mathbb{Z}; \mathbb{C}^N), \quad \langle \Psi, \Phi \rangle = \sum_{m \in \mathbb{Z}} \Psi_m^\dagger \Phi_m.$$

Let  $\mathcal{C} \subset \mathcal{H}$  denote the dense core of finitely supported sequences. Let  $\{U_m\}_{m \in \mathbb{Z}} \subset SU(N)$  be link matrices (scale-direction parallel transport). Define covariant forward/backward differences on  $\mathcal{C}$

by

$$(\nabla_+ \Psi)_m := \Psi_{m+1} - U_{m+1} \Psi_m, \quad (\nabla_- \Psi)_m := \Psi_m - U_m^\dagger \Psi_{m-1}. \quad (1)$$

On  $\mathcal{C}$  one has the adjoint relation  $\nabla_- = -\nabla_+^\dagger$  (discrete summation by parts). Define the nonnegative second-order generator

$$J := \nabla_+^\dagger \nabla_+ \geq 0, \quad (2)$$

and the symmetric first-order (scale-direction) generator

$$P_s := \frac{1}{2i} (\nabla_+ - \nabla_-). \quad (3)$$

*Remark 2.1.* In translation-invariant gauges (interior transport gauged to identity),  $P_s$  behaves as a symmetric discrete derivative and  $J$  as the discrete Laplacian. In this baseline case the Fourier symbols satisfy

$$J(k) = 4 \sin^2\left(\frac{k}{2}\right), \quad P_s(k) = \sin k, \quad P_s(k)^2 = J(k) \left(1 - \frac{J(k)}{4}\right), \quad (4)$$

equivalently  $P_s^2 = J - \frac{1}{4}J^2$  at the symbol level.

## 2.2 SU(2) holonomy normal form and the gap parameter

In the Dirac-emergence step we restrict the internal symmetry to  $SU(2)$  and work on

$$\mathcal{H}_2 := \ell^2(\mathbb{Z}; \mathbb{C}^2).$$

Let  $W \in SU(2)$  denote the unit-cell holonomy (defined up to conjugacy in one dimension). Then there exists  $\alpha \in [0, \pi]$  and a Hermitian matrix  $T$  with  $T^2 = I$  such that

$$W = \exp(i\alpha T) = \cos \alpha I + i \sin \alpha T, \quad \text{Tr}(W) = 2 \cos \alpha. \quad (5)$$

Fix a real constant  $c \in \mathbb{R}$  and define the holonomy-controlled scale gap

$$\mu := c \sin \alpha. \quad (6)$$

## 2.3 Dirac-form generator and its quadratic closure

Let  $\mathbb{C}_D^2$  denote the minimal doubled module (Dirac doublet), and set

$$\mathcal{H} := \mathcal{H}_2 \otimes \mathbb{C}_D^2.$$

Let  $\sigma_1, \sigma_2, \sigma_3$  be the Pauli matrices acting on  $\mathbb{C}_D^2$ . Assume the commutation condition on the core (holonomy axis fixed along the chain)

$$[P_s, T] = 0 \quad \text{on } \mathcal{C}. \quad (7)$$

For a real parameter  $m \in \mathbb{R}$  define the Dirac-form first-order operator on  $\mathcal{C} \otimes \mathbb{C}_D^2$  by

$$D := \sigma_1 P_s + \sigma_2 \mu T + m \sigma_3. \quad (8)$$

Using  $T^2 = I$ , (7), and the Pauli anticommutation relations, one obtains the exact quadratic closure

$$D^2 = P_s^2 + (m^2 + \mu^2) I_{\mathcal{H}} \quad \text{on } \mathcal{C} \otimes \mathbb{C}_D^2. \quad (9)$$

Define the effective gap

$$M_{\text{eff}}(\alpha) := \sqrt{m^2 + \mu^2} = \sqrt{m^2 + c^2 \sin^2 \alpha} \geq 0. \quad (10)$$

*Gauge note.* The commutation condition corresponds to the gauge in which the holonomy axis is site-independent; see Appendix C.5 for the general conjugacy justification.

## 2.4 Spectral calculus and the positive-energy branch

Since  $D$  is symmetric on the core and  $D^2$  is a nonnegative quadratic expression in  $P_s$ , we work throughout with the functional calculus induced by the self-adjoint realization of  $D$ . Define the positive operator

$$E_+ := \sqrt{D^2} = \sqrt{P_s^2 + M_{\text{eff}}^2}, \quad (11)$$

and interpret  $E_+$  as the positive-energy branch associated to the Dirac-form generator.

*Remark 2.2.* The emergence of Schrödinger dynamics will follow by (i) restricting to low-spectrum states for which  $P_s^2 \ll M_{\text{eff}}^2$ , (ii) expanding  $E_+$  uniformly on that spectral subspace, and (iii) removing the fast rest-phase  $e^{-iM_{\text{eff}}t}$  from the evolution.

## 3 Low-Spectrum Expansion and Schrödinger Limit

### 3.1 Low-spectrum spectral projection

Let  $P_s$  be self-adjoint on  $\mathcal{H}_2$  (hence  $P_s^2 \geq 0$  self-adjoint). For  $\Lambda > 0$  define the low-spectrum projection associated to  $P_s^2$  by

$$\mathbf{P}_\Lambda := \mathbf{1}_{[0, \Lambda]}(P_s^2), \quad \mathcal{H}_\Lambda := \mathbf{P}_\Lambda \mathcal{H}. \quad (12)$$

On  $\mathcal{H}_\Lambda$  one has the operator inequality

$$0 \leq P_s^2 \leq \Lambda I \quad (\text{as operators on } \mathcal{H}_\Lambda). \quad (13)$$

### 3.2 A uniform square-root expansion lemma

We isolate the analytic input needed for a controlled nonrelativistic limit.

**Lemma 3.1** (Uniform square-root expansion on a spectral window). *Fix  $M > 0$  and  $\Lambda \geq 0$ . Let  $A$  be a self-adjoint operator with  $0 \leq A \leq \Lambda I$ . Then the operator*

$$R_M(A) := \sqrt{M^2 I + A} - MI - \frac{1}{2M} A$$

*is bounded and satisfies the norm estimate*

$$\|R_M(A)\| \leq \frac{\Lambda^2}{8M^3}. \quad (14)$$

*Proof.* Consider the scalar function  $f(x) = \sqrt{M^2 + x}$  on  $[0, \Lambda]$ . It is  $C^2$  and satisfies

$$f(0) = M, \quad f'(0) = \frac{1}{2M}, \quad f''(x) = -\frac{1}{4}(M^2 + x)^{-3/2}.$$

Hence  $|f''(x)| \leq \frac{1}{4M^3}$  for all  $x \in [0, \Lambda]$ . By Taylor's theorem with remainder, for each  $x \in [0, \Lambda]$  there exists  $\xi \in (0, x)$  such that

$$f(x) = M + \frac{x}{2M} + \frac{1}{2} f''(\xi) x^2,$$

so

$$\left| f(x) - M - \frac{x}{2M} \right| \leq \frac{1}{2} \cdot \frac{1}{4M^3} x^2 \leq \frac{\Lambda^2}{8M^3}.$$

By the spectral theorem,  $\sqrt{M^2I + A} = f(A)$  and

$$R_M(A) = f(A) - \left( MI + \frac{1}{2M}A \right),$$

and the operator norm is the essential supremum of the corresponding scalar bound over  $\sigma(A) \subset [0, \Lambda]$ . This yields (14).  $\square$

### 3.3 Low-spectrum expansion of the positive-energy branch

Recall from (11) that

$$E_+ = \sqrt{P_s^2 + M_{\text{eff}}^2}, \quad M_{\text{eff}} > 0.$$

Restricting to  $\mathcal{H}_\Lambda$  and applying Lemma 3.1 with  $A = P_s^2|_{\mathcal{H}_\Lambda}$  and  $M = M_{\text{eff}}$ , we obtain the uniform expansion

$$E_+ = M_{\text{eff}}I + \frac{1}{2M_{\text{eff}}}P_s^2 + R_\Lambda, \quad \text{on } \mathcal{H}_\Lambda, \quad (15)$$

where the remainder  $R_\Lambda$  is bounded and satisfies

$$\|R_\Lambda\| \leq \frac{\Lambda^2}{8M_{\text{eff}}^3}. \quad (16)$$

### 3.4 Emergent Schrödinger generator

Define the effective Hamiltonian on  $\mathcal{H}_\Lambda$  by

$$H_{\text{eff}} := \frac{1}{2M_{\text{eff}}}P_s^2, \quad (17)$$

so that (15) reads  $E_+ = M_{\text{eff}}I + H_{\text{eff}} + R_\Lambda$  on  $\mathcal{H}_\Lambda$ .

**Theorem 3.2** (Schrödinger limit on the low-spectrum subspace). *Fix  $\Lambda > 0$  and assume  $M_{\text{eff}} > 0$ . Let  $\Psi_0 \in \mathcal{H}_\Lambda$  and define the positive-branch evolution*

$$\Psi(t) := e^{-itE_+}\Psi_0. \quad (18)$$

*Define the rest-phase-removed state*

$$\psi(t) := e^{+itM_{\text{eff}}}\Psi(t). \quad (19)$$

*Then  $\psi(t) \in \mathcal{H}_\Lambda$  for all  $t \in \mathbb{R}$  and satisfies the exact reduced evolution law*

$$i\partial_t\psi(t) = (E_+ - M_{\text{eff}}I)\psi(t). \quad (20)$$

*Moreover, on  $\mathcal{H}_\Lambda$  one has the decomposition*

$$E_+ - M_{\text{eff}}I = H_{\text{eff}} + R_\Lambda, \quad (21)$$

*with  $H_{\text{eff}}$  given by (17) and  $\|R_\Lambda\| \leq \frac{\Lambda^2}{8M_{\text{eff}}^3}$ . In particular, the reduced dynamics is Schrödinger-type:*

$$i\partial_t\psi(t) = H_{\text{eff}}\psi(t) + R_\Lambda\psi(t), \quad \text{on } \mathcal{H}_\Lambda, \quad (22)$$

*with a uniform operator-norm control on the correction term.*

*Proof.* Since  $E_+$  is self-adjoint, (18) defines a unitary group and  $\Psi(t) \in \mathcal{H}_\Lambda$  because  $\mathcal{H}_\Lambda$  is invariant under bounded Borel functions of  $P_s^2$  and  $E_+$  is a Borel function of  $P_s^2$ . Differentiating (19) gives

$$\partial_t \psi(t) = iM_{\text{eff}} e^{itM_{\text{eff}}} \Psi(t) + e^{itM_{\text{eff}}} \partial_t \Psi(t) = iM_{\text{eff}} \psi(t) - ie^{itM_{\text{eff}}} E_+ \Psi(t) = -i(E_+ - M_{\text{eff}} I) \psi(t),$$

which yields (20). The decomposition (21) and the bound on  $R_\Lambda$  are exactly (15)–(16). Substituting into (20) yields (22).  $\square$

*Remark 3.3* (Baseline lattice correction in terms of the TSFT Laplacian). In the translation-invariant baseline, (4) yields the exact identity  $P_s^2 = J - \frac{1}{4}J^2$  at the symbol level, so the effective Hamiltonian becomes

$$H_{\text{eff}} = \frac{1}{2M_{\text{eff}}} \left( J - \frac{1}{4}J^2 \right), \quad (23)$$

exhibiting a controlled lattice correction beyond the leading Laplacian term  $\frac{1}{2M_{\text{eff}}}J$ .

### 3.5 Norm conservation and unitary Schrödinger flow

We record the standard consequences of self-adjointness in a form suitable for the TSFT setting.

**Proposition 3.4** (Unitarity and norm conservation). *Let  $E_+$  be the positive operator defined by (11). Then  $e^{-itE_+}$  is unitary on  $\mathcal{H}$ , and for any  $\Psi_0 \in \mathcal{H}$  the evolution  $\Psi(t) = e^{-itE_+} \Psi_0$  satisfies*

$$\|\Psi(t)\| = \|\Psi_0\| \quad \text{for all } t \in \mathbb{R}.$$

*Moreover, for any  $\Lambda > 0$  the low-spectrum subspace  $\mathcal{H}_\Lambda$  is invariant under  $e^{-itE_+}$  and under the reduced flow  $e^{-it(E_+ - M_{\text{eff}}I)}$ .*

*Proof.* Since  $E_+$  is self-adjoint, the spectral theorem implies that  $e^{-itE_+}$  is unitary and  $\|\Psi(t)\| = \|\Psi_0\|$ . Because  $E_+$  is a Borel function of  $P_s^2$  (namely  $E_+ = \sqrt{P_s^2 + M_{\text{eff}}^2}$ ), it commutes with the spectral projection  $\mathbf{P}_\Lambda = \mathbf{1}_{[0, \Lambda]}(P_s^2)$ , so  $\mathcal{H}_\Lambda$  is invariant under the flow. The reduced flow differs only by a phase factor  $e^{itM_{\text{eff}}}$ .  $\square$

### 3.6 Quantitative comparison with the effective Schrödinger evolution

Theorem 3.2 provides a generator-level approximation. We now translate this into a uniform propagator estimate on  $\mathcal{H}_\Lambda$ .

*Long-time note.* For times comparable to or exceeding the inverse remainder scale, higher-order terms from the expansion in Appendix A may become non-negligible, and the first-order Schrödinger approximation should then be replaced by the full TSFT evolution.

**Theorem 3.5** (Propagator approximation on  $\mathcal{H}_\Lambda$ ). *Fix  $\Lambda > 0$  and assume  $M_{\text{eff}} > 0$ . Let  $\psi(t)$  be the reduced evolution defined by (19), and define the effective Schrödinger evolution*

$$\psi_{\text{eff}}(t) := e^{-itH_{\text{eff}}} \Psi_0, \quad \Psi_0 \in \mathcal{H}_\Lambda, \quad (24)$$

*with  $H_{\text{eff}} = \frac{1}{2M_{\text{eff}}}P_s^2$  as in (17). Then for all  $t \in \mathbb{R}$ ,*

$$\|\psi(t) - \psi_{\text{eff}}(t)\| \leq |t| \|R_\Lambda\| \|\Psi_0\| \leq |t| \frac{\Lambda^2}{8M_{\text{eff}}^3} \|\Psi_0\|. \quad (25)$$

*Proof.* On  $\mathcal{H}_\Lambda$  we have  $E_+ - M_{\text{eff}}I = H_{\text{eff}} + R_\Lambda$  with bounded  $R_\Lambda$ . By Duhamel's formula,

$$e^{-it(H_{\text{eff}}+R_\Lambda)} - e^{-itH_{\text{eff}}} = -i \int_0^t e^{-i(t-s)(H_{\text{eff}}+R_\Lambda)} R_\Lambda e^{-isH_{\text{eff}}} ds.$$

Taking norms and using unitarity of  $e^{-it(H_{\text{eff}}+R_\Lambda)}$  and  $e^{-itH_{\text{eff}}}$  yields

$$\|e^{-it(H_{\text{eff}}+R_\Lambda)} - e^{-itH_{\text{eff}}}\| \leq \int_0^{|t|} \|R_\Lambda\| ds = |t| \|R_\Lambda\|.$$

Applying this bound to  $\Psi_0$  gives (25), and the final estimate uses (16).  $\square$

*Remark 3.6* (Interpretation). The bound (25) shows that on the low-spectrum window  $P_s^2 \leq \Lambda$  the reduced TSFT positive-branch evolution is uniformly approximated by the Schrödinger flow generated by  $H_{\text{eff}}$  for times  $|t| \ll 8M_{\text{eff}}^3/\Lambda^2$ , with a fully explicit, model-internal error control.

### 3.7 Explicit baseline form in terms of the TSFT Laplacian

In the translation-invariant baseline, (4) yields the exact dispersion bridge  $P_s(k)^2 = J(k)(1 - J(k)/4)$ . At the operator-symbol level this is written as  $P_s^2 = J - \frac{1}{4}J^2$ , hence

$$H_{\text{eff}} = \frac{1}{2M_{\text{eff}}}\left(J - \frac{1}{4}J^2\right) = \frac{1}{2M_{\text{eff}}}J - \frac{1}{8M_{\text{eff}}}J^2. \quad (26)$$

The leading term  $\frac{1}{2M_{\text{eff}}}J$  is the standard discrete Schrödinger kinetic generator, while the  $J^2$  contribution is the controlled lattice correction intrinsic to the TSFT chain.

## 4 Discussion and Scope

### 4.1 What has been recovered and what has not

We summarize precisely what the preceding results establish and clarify the remaining gaps relative to the full structure of quantum theory.

**Standing assumption.** Throughout this work we assume the transport sequence  $\{U_m\}_{m \in \mathbb{Z}} \subset SU(N)$  is uniformly bounded (which holds automatically for unitary matrices) and that the interior gauge reduction to the holonomy normal form has been performed.

**Proposition 4.1** (Recovered nonrelativistic quantum structure within TSFT). *Within the  $SU(2)$ -covariant TSFT scale-chain framework, assume (7) and  $M_{\text{eff}} > 0$ . Then, on each low-spectrum subspace  $\mathcal{H}_\Lambda$  defined by (12):*

1. *The positive-branch dynamics generated by  $E_+ = \sqrt{P_s^2 + M_{\text{eff}}^2}$  is unitary and norm-preserving (Proposition 3.4).*
2. *After removing the fast rest-phase  $e^{-iM_{\text{eff}}t}$ , the reduced evolution obeys an exact first-order law (20).*
3. *The reduced generator admits the uniform low-spectrum decomposition  $E_+ - M_{\text{eff}}I = H_{\text{eff}} + R_\Lambda$  with  $H_{\text{eff}} = \frac{1}{2M_{\text{eff}}}P_s^2$  and an explicit error bound  $\|R_\Lambda\| \leq \Lambda^2/(8M_{\text{eff}}^3)$  (Theorem 3.2).*

4. *The reduced propagator is uniformly approximated by the Schrödinger flow  $e^{-itH_{\text{eff}}}$  with an explicit Duhamel estimate (25) (Theorem 3.5).*
5. *In the translation-invariant baseline,  $H_{\text{eff}}$  admits an explicit Laplacian-plus-correction form (26) in terms of the TSFT second-order operator  $J$ .*

*Remark 4.2* (Relation to the TSFT spectral sequence). The present paper supplies the dynamical complement to the earlier kinematic backbone: (i) covariant self-adjoint construction and holonomy-controlled Floquet sectorization, (ii) boundary/half-line admissibility yielding Bohr-type discrete modes, and (iii) Weyl-pair scale–phase conjugacy yielding a Heisenberg-type uncertainty structure. Here we show that the Dirac-form first-order closure also implies a Schrödinger-type evolution on the low-spectrum sector, with controlled lattice corrections.

## 4.2 Scope limitations

The results are structural and do not, by themselves, constitute a full derivation of all of quantum mechanics. In particular:

1. **Born rule and measurement.** We have derived a unitary evolution and an effective Schrödinger generator on a low-spectrum sector, but no probabilistic measurement postulate or Born-rule derivation is included.
2. **Interactions.** The analysis is single-particle/linear-operator in spirit. Gauge curvature dynamics, interaction terms, and multi-body coupling are not introduced here.
3. **Particle identification.** No Standard Model fermions are identified; the holonomy parameter  $\alpha$  enters only through the conjugacy invariant  $\text{Tr}(W) = 2 \cos \alpha$  and the effective gap  $M_{\text{eff}}(\alpha)$ .

While Schrödinger-type unitary evolution can arise from general self-adjoint generators, the TSFT construction is structurally constrained by  $\text{SU}(N)$ -covariant scale geometry and produces specific holonomy-controlled corrections (e.g., the  $J^2$  term) not present in generic lattice discretizations.

## 4.3 Next steps

The operator-theoretic bridge established here suggests several immediate continuations:

1. **Sector selection under drift and boundary compatibility.** Integrate the Schrödinger limit with the half-line admissibility mechanism to obtain a discrete ladder of effective non-relativistic energies tied directly to monodromy closure.
2. **Continuum and multi-dimensional extensions.** Extend the present derivation to higher-dimensional lattices and/or continuum scale variables, to connect  $J$  more directly to standard Laplacians and to clarify the dispersion corrections.
3. **Interaction channels.** Explore minimal TSFT-consistent perturbations (still self-adjoint at the quadratic form level) that generate interaction terms while preserving the forced algebraic closures of the spectral sequence.

## Acknowledgments

The author thanks the ongoing TSFT development and review process within The Zebra Journal of Unified Physics (ZJUP).

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## A Technical Operator Analysis of the Low-Spectrum Limit

This appendix collects the functional-analytic details underlying the low-spectrum Schrödinger reduction proved in the main text. Throughout we assume the standing hypotheses of Section 2, in particular that  $P_s$  is self-adjoint and  $M_{\text{eff}} > 0$ .

### A.1 Spectral calculus for the Dirac square

Recall the exact quadratic closure

$$D^2 = P_s^2 + M_{\text{eff}}^2 I, \quad M_{\text{eff}} > 0. \quad (27)$$

**Lemma A.1** (Self-adjoint functional calculus). *Let  $P_s$  be self-adjoint on  $\mathcal{H}_2$ . Then  $D^2$  defined by (27) is self-adjoint and strictly positive, and the operator*

$$E_+ := \sqrt{D^2}$$

*is self-adjoint with domain given by the spectral calculus of  $P_s^2$ .*

*Proof.* Since  $P_s$  is self-adjoint,  $P_s^2$  is self-adjoint and nonnegative. Adding the strictly positive constant  $M_{\text{eff}}^2 I$  preserves self-adjointness and yields the lower bound

$$D^2 \geq M_{\text{eff}}^2 I > 0.$$

The square-root functional calculus for nonnegative self-adjoint operators (Reed–Simon I, Thm. VIII.6) yields a unique self-adjoint operator  $E_+ = \sqrt{D^2}$ .  $\square$

### A.2 Precise low-spectrum control

We now quantify the small-parameter regime used in the main theorem.

**Lemma A.2** (Low-spectrum domination). *On the spectral subspace  $\mathcal{H}_\Lambda$  defined by*

$$\mathcal{H}_\Lambda = \mathbf{1}_{[0,\Lambda]}(P_s^2)\mathcal{H},$$

*one has the operator inequality*

$$\frac{P_s^2}{M_{\text{eff}}^2} \leq \frac{\Lambda}{M_{\text{eff}}^2} I.$$

*In particular, if  $\Lambda < M_{\text{eff}}^2$ , then the dimensionless operator*

$$X := M_{\text{eff}}^{-2} P_s^2$$

*satisfies  $\|X\| \leq \Lambda/M_{\text{eff}}^2 < 1$  on  $\mathcal{H}_\Lambda$ .*

*Proof.* Immediate from the spectral bound  $0 \leq P_s^2 \leq \Lambda I$  on  $\mathcal{H}_\Lambda$ . □

### A.3 Operator Taylor expansion with remainder

We now give a fully explicit operator expansion used implicitly in the main text.

**Theorem A.3** (Operator square-root expansion). *Let  $A \geq 0$  be self-adjoint with  $\|A\| \leq \Lambda$ , and let  $M > 0$ . Then on the whole Hilbert space,*

$$\sqrt{M^2 I + A} = MI + \frac{1}{2M}A - \frac{1}{8M^3}A^2 + \mathcal{R}_3(A), \quad (28)$$

*where the remainder satisfies the operator-norm bound*

$$\|\mathcal{R}_3(A)\| \leq \frac{\Lambda^3}{16M^5}. \quad (29)$$

*Proof.* Let  $f(x) = \sqrt{M^2 + x}$  on  $[0, \Lambda]$ . A third-order Taylor expansion with integral remainder gives

$$f(x) = M + \frac{x}{2M} - \frac{x^2}{8M^3} + \frac{f^{(3)}(\xi_x)}{6}x^3$$

for some  $\xi_x \in (0, x)$ .

Direct differentiation yields

$$f^{(3)}(x) = \frac{3}{8}(M^2 + x)^{-5/2},$$

hence

$$|f^{(3)}(x)| \leq \frac{3}{8M^5} \quad \text{for } x \in [0, \Lambda].$$

Therefore

$$\left| \frac{f^{(3)}(\xi_x)}{6}x^3 \right| \leq \frac{1}{16M^5}x^3 \leq \frac{\Lambda^3}{16M^5}.$$

The operator statement follows from the spectral theorem applied to  $A$ . □

## A.4 Application to the TSFT Dirac branch

We now specialize Theorem A.3 to the TSFT setting.

**Corollary A.4** (TSFT positive-branch expansion). *On  $\mathcal{H}_\Lambda$  one has*

$$E_+ = M_{\text{eff}}I + \frac{1}{2M_{\text{eff}}}P_s^2 - \frac{1}{8M_{\text{eff}}^3}P_s^4 + \mathcal{R}_3(P_s^2), \quad (30)$$

$$\|\mathcal{R}_3(P_s^2)\| \leq \frac{\Lambda^3}{16M_{\text{eff}}^5}. \quad (31)$$

*Proof.* Apply Theorem A.3 with  $A = P_s^2$  and  $M = M_{\text{eff}}$ , using  $\|P_s^2\| \leq \Lambda$  on  $\mathcal{H}_\Lambda$ .  $\square$

## A.5 Generator-level error estimate

**Proposition A.5** (Effective Hamiltonian accuracy). *Let*

$$H_{\text{eff}} = \frac{1}{2M_{\text{eff}}}P_s^2.$$

*Then on  $\mathcal{H}_\Lambda$ ,*

$$\|(E_+ - M_{\text{eff}}I) - H_{\text{eff}}\| \leq \frac{\Lambda^2}{8M_{\text{eff}}^3}. \quad (32)$$

*Proof.* This is the first-order truncation of Corollary A.4 together with  $\|P_s^4\| \leq \Lambda^2$  on  $\mathcal{H}_\Lambda$ .  $\square$

## A.6 Continuum long-wavelength correspondence

Finally we record the formal continuum bridge used in the discussion.

**Lemma A.6** (Long-wavelength symbol reduction). *In the translation-invariant baseline, the Fourier symbol of  $J$  satisfies*

$$J(k) = k^2 + O(k^4) \quad \text{as } k \rightarrow 0,$$

*and therefore*

$$P_s^2(k) = k^2 + O(k^4).$$

*Proof.* Using  $J(k) = 4\sin^2(k/2)$  and  $\sin(k/2) = k/2 + O(k^3)$  gives

$$J(k) = k^2 + O(k^4).$$

Substituting into  $P_s^2 = J - \frac{1}{4}J^2$  yields the claim.  $\square$

*Remark A.7.* Lemma A.6 explains why the leading term of  $H_{\text{eff}}$  reproduces the standard nonrelativistic kinetic energy in the long-wavelength regime while retaining controlled lattice corrections at higher momentum.

## B Self-Adjointness and Domain Analysis

This appendix provides a rigorous operator-theoretic foundation for the generators used in the main text. We verify symmetry, essential self-adjointness on the core, and the legitimacy of the functional calculus applied to the Dirac-form operator.

Throughout, we work on the Hilbert space

$$\mathcal{H} := \ell^2(\mathbb{Z}; \mathbb{C}^N),$$

with dense core

$$\mathcal{C} := \{\Psi \in \mathcal{H} : \Psi \text{ has finite support}\}.$$

### B.1 Symmetry of the covariant differences

Recall the covariant differences

$$(\nabla_+ \Psi)_m = \Psi_{m+1} - U_{m+1} \Psi_m, \quad (\nabla_- \Psi)_m = \Psi_m - U_m^\dagger \Psi_{m-1},$$

with  $U_m \in SU(N)$ .

**Lemma B.1** (Discrete summation by parts). *On the core  $\mathcal{C}$  one has*

$$\nabla_- = -\nabla_+^\dagger.$$

*Proof.* Let  $\Psi, \Phi \in \mathcal{C}$ . Using finite support and unitarity of  $U_m$ ,

$$\begin{aligned} \langle \nabla_+ \Psi, \Phi \rangle &= \sum_m (\Psi_{m+1} - U_{m+1} \Psi_m)^\dagger \Phi_m \\ &= \sum_m \Psi_{m+1}^\dagger \Phi_m - \sum_m \Psi_m^\dagger U_{m+1}^\dagger \Phi_m. \end{aligned}$$

Shift the first sum  $m \mapsto m - 1$  and use finite support:

$$\sum_m \Psi_{m+1}^\dagger \Phi_m = \sum_m \Psi_m^\dagger \Phi_{m-1}.$$

Hence

$$\langle \nabla_+ \Psi, \Phi \rangle = \sum_m \Psi_m^\dagger (\Phi_{m-1} - U_{m+1}^\dagger \Phi_m) = -\langle \Psi, \nabla_- \Phi \rangle.$$

□

**Proposition B.2** (Symmetry of  $P_s$ ). *The operator*

$$P_s = \frac{1}{2i} (\nabla_+ - \nabla_-)$$

*is symmetric on the core  $\mathcal{C}$ .*

*Proof.* Immediate from Lemma B.1.

□

## B.2 Essential self-adjointness of $P_s$

We now justify that  $P_s$  admits a unique self-adjoint realization.

**Theorem B.3** (Essential self-adjointness of the scale generator). *Assume the link sequence  $\{U_m\}$  is uniformly bounded and unitary (as holds for  $U_m \in SU(N)$ ). Then the symmetric operator  $P_s$  defined on  $\mathcal{C}$  is essentially self-adjoint on  $\ell^2(\mathbb{Z}; \mathbb{C}^N)$ .*

*Proof.* The operator  $P_s$  is a finite-band difference operator with uniformly bounded matrix coefficients. Such operators on  $\ell^2(\mathbb{Z}; \mathbb{C}^N)$  are essentially self-adjoint on the finitely supported core (see, e.g., Reed–Simon II, Thm. X.26; Teschl, Jacobi Operators).

More concretely,  $P_s$  is a bounded perturbation of the free discrete derivative (obtained when  $U_m = I$ ), which is known to be essentially self-adjoint on  $\mathcal{C}$ . Bounded perturbations preserve essential self-adjointness (Kato–Rellich theorem), yielding the claim.  $\square$

**Corollary B.4** (Self-adjoint square). *The operator  $P_s^2$  defined via the spectral calculus of the closure of  $P_s$  is self-adjoint and nonnegative.*

*Proof.* Standard functional calculus for self-adjoint operators.  $\square$

## B.3 Self-adjointness of the Dirac-form operator

Recall the Dirac-form operator

$$D = \sigma_1 P_s + \sigma_2 \mu T + m \sigma_3,$$

acting on

$$\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{C}^2) \otimes \mathbb{C}_D^2,$$

with  $T = T^\dagger$  and  $T^2 = I$ .

**Theorem B.5** (Self-adjointness of  $D$ ). *Assume  $[P_s, T] = 0$  on the core and  $\mu, m \in \mathbb{R}$ . Then the operator  $D$  is essentially self-adjoint on  $\mathcal{C} \otimes \mathbb{C}_D^2$ .*

*Proof.* By Theorem B.3,  $P_s$  is essentially self-adjoint. The remaining terms  $\sigma_2 \mu T$  and  $m \sigma_3$  are bounded self-adjoint operators on  $\mathcal{H}$ .

Hence  $D$  is a bounded self-adjoint perturbation of the essentially self-adjoint operator  $\sigma_1 P_s$ . The Kato–Rellich theorem implies essential self-adjointness.  $\square$

**Corollary B.6** (Positive branch well-defined). *The operator*

$$E_+ = \sqrt{D^2}$$

*is self-adjoint and generates a strongly continuous unitary group  $e^{-itE_+}$  on  $\mathcal{H}$ .*

*Proof.* Since  $D$  is self-adjoint,  $D^2$  is self-adjoint and nonnegative. The square-root functional calculus yields self-adjoint  $E_+$ . Stone’s theorem gives the unitary group.  $\square$

## B.4 Domain invariance of the low-spectrum sector

**Proposition B.7** (Spectral subspace invariance). *Let*

$$\mathcal{H}_\Lambda = \mathbf{1}_{[0,\Lambda]}(P_s^2)\mathcal{H}.$$

*Then  $\mathcal{H}_\Lambda$  is invariant under*

1.  $e^{-itE_+}$ ,
2.  $e^{-it(E_+ - M_{\text{eff}}I)}$ ,
3.  $e^{-itH_{\text{eff}}}$ .

*Proof.* Each generator is a bounded Borel function of  $P_s^2$ , hence commutes with the spectral projection  $\mathbf{1}_{[0,\Lambda]}(P_s^2)$ .  $\square$

*Remark B.8* (Technical conclusion). The TSFT scale-chain generators used in the main text admit a fully controlled self-adjoint realization. All functional calculi and unitary evolutions invoked in the Schrödinger limit are therefore mathematically well-posed.

## C SU(2) Holonomy Normal Form and Gauge Reduction

This appendix provides the group-theoretic justification for the holonomy parametrization used in the main text. We show that any unit-cell holonomy in  $SU(2)$  is conjugate to a one-parameter involution form and that the associated generator  $T$  may be chosen Hermitian with  $T^2 = I$ . This reduction is canonical up to unitary conjugacy and underlies the effective gap parameter

$$\mu = c \sin \alpha.$$

### C.1 Spectral structure of SU(2)

**Lemma C.1** (Eigenvalue structure of  $SU(2)$ ). *Let  $W \in SU(2)$ . Then:*

1. *The eigenvalues of  $W$  are  $\{e^{i\alpha}, e^{-i\alpha}\}$  for some  $\alpha \in [0, \pi]$ .*
2. *The trace determines  $\alpha$  uniquely via*

$$\text{Tr}(W) = 2 \cos \alpha.$$

*Proof.* Since  $W$  is unitary, its eigenvalues lie on the unit circle. Because  $\det W = 1$ , the eigenvalues are complex conjugates  $\lambda, \bar{\lambda}$  with  $|\lambda| = 1$ , so we may write  $\lambda = e^{i\alpha}$  for some  $\alpha \in [0, \pi]$ . The trace relation follows immediately.  $\square$

### C.2 Conjugacy classification

**Theorem C.2** ( $SU(2)$  conjugacy normal form). *For any  $W \in SU(2)$  there exists a unitary matrix  $V \in SU(2)$  such that*

$$VWV^\dagger = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \quad \alpha \in [0, \pi]. \quad (33)$$

*Proof.* Because  $W$  is normal, it is unitarily diagonalizable. The determinant-one constraint fixes the eigenvalue pair to be conjugate reciprocals.  $\square$

*Remark C.3.* Equation (33) shows that the conjugacy class of  $W$  is completely determined by  $\alpha$ , equivalently by  $\text{Tr}(W)$ .

### C.3 Involution representation

We now convert the diagonal form into the involutive structure used in the Dirac reduction.

**Theorem C.4** (Hermitian involution representation). *Let  $W \in SU(2)$  with parameter  $\alpha$  as above. Then there exists a Hermitian matrix  $T$  such that*

$$T^\dagger = T, \quad T^2 = I,$$

and

$$W = \exp(i\alpha T) = \cos \alpha I + i \sin \alpha T. \quad (34)$$

*Proof.* In the diagonal gauge (33), define

$$T_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then  $T_0^\dagger = T_0$  and  $T_0^2 = I$ , and

$$\exp(i\alpha T_0) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}.$$

Returning to the original gauge, set

$$T = V^\dagger T_0 V.$$

Then  $T$  remains Hermitian with  $T^2 = I$ , and

$$W = V^\dagger \exp(i\alpha T_0) V = \exp(i\alpha T).$$

The cosine–sine decomposition follows from the functional calculus of the involution  $T$ . □

### C.4 Uniqueness up to conjugacy

**Proposition C.5** (Gauge freedom). *If  $T$  satisfies (34), then any unitary conjugate  $UTU^\dagger$  yields the same holonomy  $W$  up to global conjugation. Thus the pair  $(\alpha, T)$  is unique modulo unitary gauge transformations.*

*Proof.* Immediate from functional calculus and unitary covariance of the exponential map. □

*Remark C.6.* All gauge-invariant spectral information depends only on  $\alpha$  (equivalently  $\text{Tr}(W)$ ), not on the specific choice of  $T$ .

### C.5 Commutation with the scale generator

The Dirac reduction in the main text assumes

$$[P_s, T] = 0 \quad \text{on the core.}$$

**Lemma C.7** (Gauge alignment condition). *If the holonomy axis is constant along the scale chain, one may choose the gauge so that  $T$  is site-independent. In this gauge  $T$  commutes with the covariant difference operator and hence with  $P_s$ .*

*Proof.* When the transport links are gauge-reduced to identity in the interior (holonomy normal form of the periodic chain), the remaining internal action is constant along the lattice. In this gauge  $T$  acts only on the internal  $\mathbb{C}^2$  factor and therefore commutes with the shift/difference structure defining  $P_s$ .  $\square$

*Remark C.8* (Role in Dirac emergence). The commutation condition ensures cancellation of cross terms in  $D^2$  and is the algebraic mechanism forcing the relativistic quadratic closure used in the main text.

## C.6 Effective gap parameter

From the representation (34), the TSFT gap parameter is

$$\mu = c \sin \alpha, \quad M_{\text{eff}} = \sqrt{m^2 + \mu^2}.$$

**Proposition C.9** (Holonomy control of the gap). *The effective mass scale  $M_{\text{eff}}$  depends only on the conjugacy invariant  $\text{Tr}(W)$  and is therefore gauge-invariant.*

*Proof.* By Lemma C.1,  $\alpha$  is determined by  $\text{Tr}(W)$ , and the formula for  $M_{\text{eff}}$  depends only on  $\alpha$ .  $\square$

*Remark C.10* (Structural role). Appendix C establishes that the Dirac-type mass gap in the TSFT chain is not imposed externally but is a direct consequence of the  $\text{SU}(2)$  holonomy class of the periodic transport structure.

## D Discrete-to-Continuum Correspondence

This appendix makes precise the long-wavelength bridge between the TSFT scale-chain operators and the standard Schrödinger kinetic operator. We introduce an explicit lattice spacing parameter and derive the continuum limit under controlled scaling.

Throughout this appendix we work in the translation-invariant baseline gauge where interior transport has been reduced to the identity, so that the core discrete structure is exposed.

### D.1 Lattice spacing and rescaled operators

Let  $a > 0$  denote the physical lattice spacing. We associate the site index  $m \in \mathbb{Z}$  with the spatial coordinate

$$x_m := am.$$

Define the rescaled forward/backward differences on  $\ell^2(\mathbb{Z})$ :

$$(\delta_+^{(a)} \Psi)_m := \frac{\Psi_{m+1} - \Psi_m}{a}, \tag{35}$$

$$(\delta_-^{(a)} \Psi)_m := \frac{\Psi_m - \Psi_{m-1}}{a}. \tag{36}$$

The corresponding discrete Laplacian is

$$(\Delta_a \Psi)_m := \delta_-^{(a)} \delta_+^{(a)} \Psi_m = \frac{\Psi_{m+1} - 2\Psi_m + \Psi_{m-1}}{a^2}. \tag{37}$$

In the baseline gauge, the TSFT operators satisfy

$$P_s^{(a)} \sim -i \delta^{(a)}, \quad J^{(a)} \sim -\Delta_a,$$

up to higher-order lattice corrections.

## D.2 Fourier symbol analysis

Let  $\widehat{\Psi}(k)$  denote the discrete Fourier transform. The symbols of the rescaled operators are:

$$\widehat{\delta_+^{(a)}}(k) = \frac{e^{ika} - 1}{a}, \quad (38)$$

$$\widehat{\Delta_a}(k) = \frac{2 \cos(ka) - 2}{a^2} = -\frac{4}{a^2} \sin^2\left(\frac{ka}{2}\right). \quad (39)$$

**Lemma D.1** (Long-wavelength expansion). *As  $a \rightarrow 0$  with  $k$  fixed,*

$$\widehat{\Delta_a}(k) = -k^2 + O(a^2 k^4), \quad (40)$$

$$\widehat{\delta_+^{(a)}}(k) = ik + O(a^2 k^3). \quad (41)$$

*Proof.* Use the Taylor expansions

$$\sin\left(\frac{ka}{2}\right) = \frac{ka}{2} + O(a^3 k^3), \quad e^{ika} = 1 + ika - \frac{k^2 a^2}{2} + O(a^3 k^3).$$

Substitution yields the stated asymptotics.  $\square$

## D.3 Continuum limit of the TSFT kinetic term

Recall the effective Hamiltonian

$$H_{\text{eff}} = \frac{1}{2M_{\text{eff}}} P_s^2.$$

In the baseline symbol calculus (Section 4) we have

$$P_s^2 = J - \frac{1}{4} J^2.$$

Introduce the rescaled operator  $J^{(a)}$  whose symbol is

$$J^{(a)}(k) = \frac{4}{a^2} \sin^2\left(\frac{ka}{2}\right).$$

**Theorem D.2** (Continuum Schrödinger limit). *Let  $\psi_a$  be a family of states whose Fourier support satisfies  $|k| \leq K$  uniformly in  $a$ . Then as  $a \rightarrow 0$ ,*

$$H_{\text{eff}}^{(a)} \psi_a = -\frac{1}{2M_{\text{eff}}} \partial_x^2 \psi + O(a^2), \quad (42)$$

*in the strong operator sense on such low-momentum states.*

*Proof.* Using Lemma D.1,

$$J^{(a)}(k) = k^2 + O(a^2 k^4).$$

Therefore

$$(J^{(a)})^2(k) = k^4 + O(a^2 k^6).$$

Substituting into

$$P_s^2 = J - \frac{1}{4} J^2$$

gives

$$P_s^{(a)2}(k) = k^2 + O(a^2 k^4).$$

Multiplying by  $(2M_{\text{eff}})^{-1}$  yields the claim.  $\square$

## D.4 Time-evolution correspondence

We now connect the discrete evolution to the continuum Schrödinger equation.

**Corollary D.3** (Continuum time evolution). *Let  $\psi_a(t)$  solve*

$$i\partial_t\psi_a = H_{\text{eff}}^{(a)}\psi_a$$

*with uniformly low-momentum initial data. Then as  $a \rightarrow 0$ ,  $\psi_a$  converges (on compact time intervals) to the solution of*

$$i\partial_t\psi = -\frac{1}{2M_{\text{eff}}}\partial_x^2\psi.$$

*Proof.* Combine Theorem D.2 with standard strong convergence of generators for uniformly bounded low-momentum states (Trotter–Kato theorem).  $\square$

## D.5 Interpretation

*Remark D.4* (Physical meaning). Appendix D shows that the TSFT scale-chain does not merely mimic Schrödinger dynamics formally: in the long-wavelength sector the effective generator converges to the standard nonrelativistic kinetic operator with explicitly controlled  $O(a^2)$  lattice corrections.

*Remark D.5* (Regime of validity). The continuum reduction requires

- low spectral window  $P_s^2 \ll M_{\text{eff}}^2$  (nonrelativistic regime),
- long wavelength  $|k|a \ll 1$ ,
- fixed holonomy parameter  $\alpha$ .

Outside this regime the full TSFT dispersion must be retained.

## E Parameter Regime Map and Scaling Structure

This appendix summarizes the hierarchy of dynamical regimes implicit in the TSFT scale-chain and provides quantitative criteria for the validity of the Schrödinger reduction. The goal is to make explicit the separation between nonrelativistic, relativistic, and lattice-dominated behavior.

Throughout we assume the standing hypotheses of the main text and define

$$E_+(k) = \sqrt{P_s^2(k) + M_{\text{eff}}^2}, \quad M_{\text{eff}} = \sqrt{m^2 + c^2 \sin^2 \alpha}.$$

### E.1 Dimensionless control parameters

The behavior of the TSFT dynamics is governed by two independent small parameters.

[Nonrelativistic parameter] The spectral (velocity) parameter is

$$\varepsilon(k) := \frac{P_s^2(k)}{M_{\text{eff}}^2}. \tag{43}$$

[Lattice parameter] With lattice spacing  $a > 0$ , define

$$\eta(k) := |k|a. \tag{44}$$

*Remark E.1.* The Schrödinger regime corresponds to the joint limit

$$\varepsilon \ll 1, \quad \eta \ll 1.$$

## E.2 Regime classification

**Proposition E.2** (Dynamical regime map). *Let  $k$  denote the Fourier variable. The TSFT dynamics separates into the following regimes:*

1. *Nonrelativistic Schrödinger regime*

$$P_s^2(k) \ll M_{\text{eff}}^2 \quad \text{and} \quad |k|a \ll 1.$$

2. *Relativistic (Dirac) regime*

$$P_s^2(k) \sim M_{\text{eff}}^2, \quad |k|a \ll 1.$$

3. *Lattice-dominated regime*

$$|k|a \sim 1,$$

where higher-order symbol corrections become non-negligible.

4. *Ultraviolet TSFT regime*

$$|k|a \gg 1,$$

where the full discrete dispersion must be retained.

*Proof.* Immediate from the dispersion relation and the long-wavelength expansion in Appendix D.  $\square$

## E.3 Energy expansion across regimes

**Lemma E.3** (Uniform energy expansion). *For  $\varepsilon(k) < 1$  one has the convergent expansion*

$$E_+(k) = M_{\text{eff}} + \frac{P_s^2(k)}{2M_{\text{eff}}} - \frac{P_s^4(k)}{8M_{\text{eff}}^3} + O\left(\frac{P_s^6(k)}{M_{\text{eff}}^5}\right). \quad (45)$$

*Proof.* This follows from the analytic square-root expansion applied to  $E_+(k) = M_{\text{eff}}\sqrt{1 + \varepsilon(k)}$ .  $\square$

*Remark E.4.* The Schrödinger approximation corresponds to truncating after the first correction term.

## E.4 Breakdown scale of the Schrödinger approximation

**Theorem E.5** (Quantitative validity window). *Fix an error tolerance  $\delta > 0$ . The relative error of the Schrödinger approximation satisfies*

$$\frac{|E_+(k) - (M_{\text{eff}} + P_s^2(k)/(2M_{\text{eff}}))|}{M_{\text{eff}}} \leq \frac{\varepsilon(k)^2}{8}. \quad (46)$$

*In particular, the nonrelativistic approximation is accurate to order  $\delta$  whenever*

$$\varepsilon(k) \leq \sqrt{8\delta}.$$

*Proof.* Directly from the second-order Taylor remainder for the square root.  $\square$

## E.5 Holonomy dependence of the regimes

**Proposition E.6** (Holonomy control of the scale hierarchy). *The effective mass scale*

$$M_{\text{eff}}(\alpha) = \sqrt{m^2 + c^2 \sin^2 \alpha}$$

*monotonically increases with  $|\sin \alpha|$ . Consequently:*

1. *Larger holonomy angle  $\alpha$  enlarges the nonrelativistic window.*
2. *Small  $\alpha$  compresses the Schrödinger regime and pushes the system toward relativistic behavior.*

*Proof.* Immediate from differentiation of  $M_{\text{eff}}(\alpha)$ . □

## E.6 Time-scale separation

**Lemma E.7** (Rest-phase separation scale). *The fast oscillation removed in the main text has frequency  $M_{\text{eff}}$ . The Schrödinger time scale is governed by*

$$\tau_{\text{NR}} \sim \frac{M_{\text{eff}}}{P_s^2}.$$

*Hence in the regime  $P_s^2 \ll M_{\text{eff}}^2$  one has*

$$\tau_{\text{NR}} \gg M_{\text{eff}}^{-1},$$

*establishing a clean two-time-scale structure.*

*Proof.* Immediate from the expansion of  $E_+(k)$ . □

## E.7 Summary diagram (conceptual)

*Remark E.8* (Hierarchy of scales). The TSFT spectral geometry exhibits the ordered structure

$$\text{UV lattice} \longrightarrow \text{Dirac regime} \longrightarrow \text{Schrödinger regime},$$

controlled respectively by the parameters

$$|k|a, \quad \frac{P_s^2}{M_{\text{eff}}^2}, \quad \frac{P_s^2}{M_{\text{eff}}^2} \ll 1.$$

This hierarchy explains why nonrelativistic quantum dynamics emerges as a robust infrared limit of the TSFT chain.

**Lemma E.9** (Commutation for functions of a self-adjoint operator). *Let  $A$  be a self-adjoint operator on a Hilbert space  $\mathcal{K}$ , and let  $f, g : \mathbb{R} \rightarrow \mathbb{C}$  be bounded Borel functions. Then the bounded operators  $f(A)$  and  $g(A)$  commute:*

$$f(A)g(A) = g(A)f(A).$$

*Moreover, if  $P := \mathbf{1}_B(A)$  is a spectral projection of  $A$  for some Borel set  $B \subset \mathbb{R}$ , then  $P$  commutes with  $f(A)$  and the range  $\text{Ran}(P)$  is invariant under  $f(A)$ .*

*Proof.* By the spectral theorem, there exists a projection-valued measure  $E_A(\cdot)$  such that

$$f(A) = \int_{\mathbb{R}} f(\lambda) dE_A(\lambda), \quad g(A) = \int_{\mathbb{R}} g(\lambda) dE_A(\lambda).$$

For bounded Borel functions, the functional calculus is multiplicative:

$$f(A)g(A) = (fg)(A) = (gf)(A) = g(A)f(A),$$

which yields commutation. For the spectral projection  $P = \mathbf{1}_B(A)$ , we have  $P = f_0(A)$  with  $f_0 = \mathbf{1}_B$ , so the commutation follows from the first part. Finally,  $Pf(A) = f(A)P$  implies  $f(A)\text{Ran}(P) \subseteq \text{Ran}(P)$ , so  $\text{Ran}(P)$  is invariant.  $\square$

**Lemma E.10** (Boundedness and domain stability on a low-spectrum subspace). *Let  $A \geq 0$  be self-adjoint on a Hilbert space  $\mathcal{K}$  and fix  $\Lambda > 0$ . Let*

$$P_\Lambda := \mathbf{1}_{[0,\Lambda]}(A), \quad \mathcal{K}_\Lambda := \text{Ran}(P_\Lambda).$$

*Then:*

1. *The restriction  $A|_{\mathcal{K}_\Lambda}$  is bounded and satisfies*

$$\|A|_{\mathcal{K}_\Lambda}\| \leq \Lambda.$$

*In particular,  $\mathcal{K}_\Lambda \subset \text{Dom}(A)$  and, more generally,*

$$\mathcal{K}_\Lambda \subset \text{Dom}(A^n) \quad \text{for every } n \in \mathbb{N}.$$

2. *For any Borel function  $f$  finite on  $[0, \Lambda]$  (in particular any bounded Borel  $f$ ), the operator  $f(A)$  leaves  $\mathcal{K}_\Lambda$  invariant:*

$$f(A)\mathcal{K}_\Lambda \subseteq \mathcal{K}_\Lambda.$$

3. *If  $f$  is bounded on  $[0, \Lambda]$ , then  $f(A)|_{\mathcal{K}_\Lambda}$  is bounded and*

$$\|f(A)|_{\mathcal{K}_\Lambda}\| \leq \sup_{\lambda \in [0,\Lambda]} |f(\lambda)|.$$

*Proof.* By the spectral theorem, the spectrum of  $A|_{\mathcal{K}_\Lambda}$  is contained in  $[0, \Lambda]$ , hence  $\|A|_{\mathcal{K}_\Lambda}\| \leq \Lambda$ , proving (1). In particular, for  $x \in \mathcal{K}_\Lambda$ ,

$$\|A^n x\|^2 = \int_{[0,\Lambda]} \lambda^{2n} d\langle E_A(\lambda)x, x \rangle \leq \Lambda^{2n} \|x\|^2,$$

so  $x \in \text{Dom}(A^n)$  for all  $n$ .

For (2), note that  $P_\Lambda = \mathbf{1}_{[0,\Lambda]}(A)$  is a function of  $A$ , so by Lemma E.9 it commutes with  $f(A)$ . Thus

$$f(A)\mathcal{K}_\Lambda = f(A)\text{Ran}(P_\Lambda) \subseteq \text{Ran}(P_\Lambda) = \mathcal{K}_\Lambda.$$

For (3), restrict the functional calculus to the spectral measure on  $[0, \Lambda]$ :

$$\|f(A)x\|^2 = \int_{[0,\Lambda]} |f(\lambda)|^2 d\langle E_A(\lambda)x, x \rangle \leq \left( \sup_{\lambda \in [0,\Lambda]} |f(\lambda)| \right)^2 \|x\|^2,$$

for all  $x \in \mathcal{K}_\Lambda$ , giving the bound.  $\square$